Logical Translations

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1 Examples

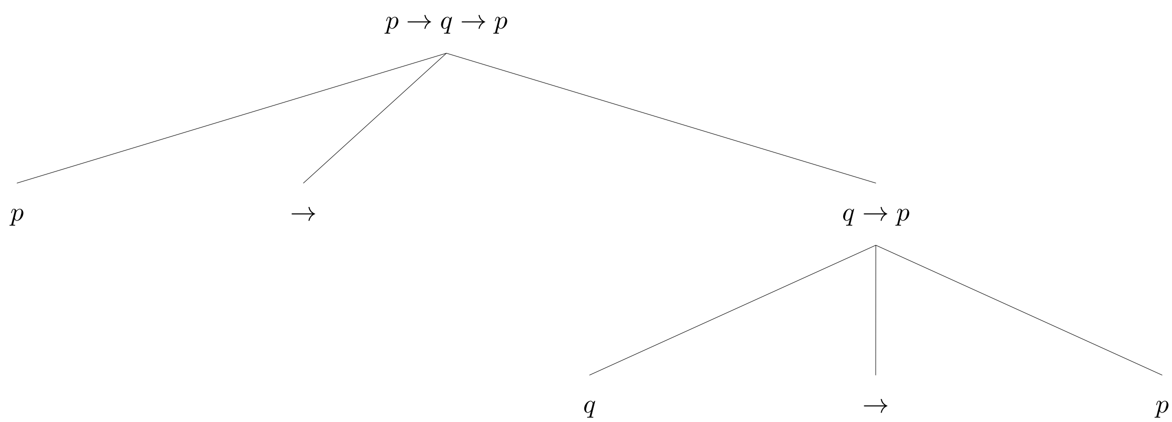


Figure 1: Logic tree for $p \to (q \to p)$

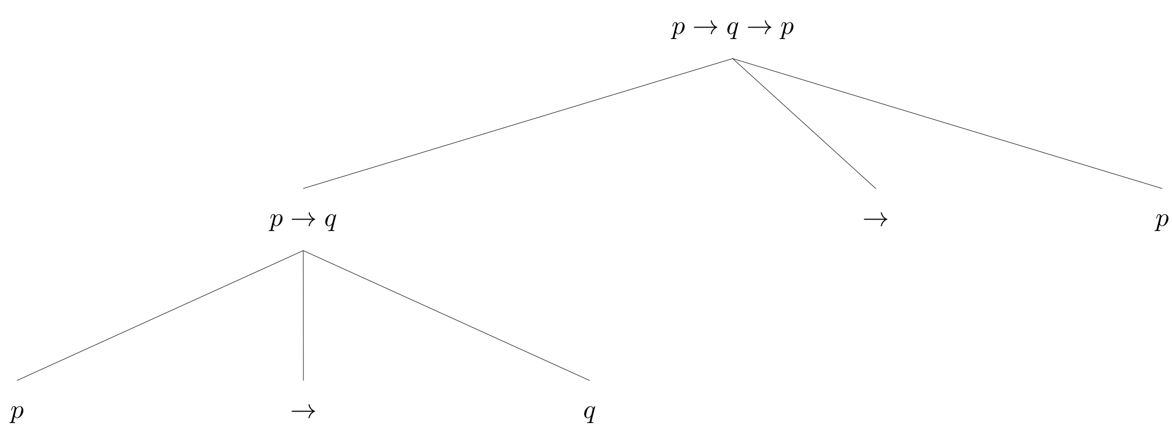


Figure 2: Logic tree for $(p \to q) \to p$

	p	q	$p \to q$
(a)	T	T	T
(b)	T	F	F
(c)	F	T	T
(d)	F	F	T

Truth table for $p \to q$

	p	q
w_1	T	T
w_2	T	F
w_3	F	T
w_4	F	F

Truth of p and q in 4 worlds

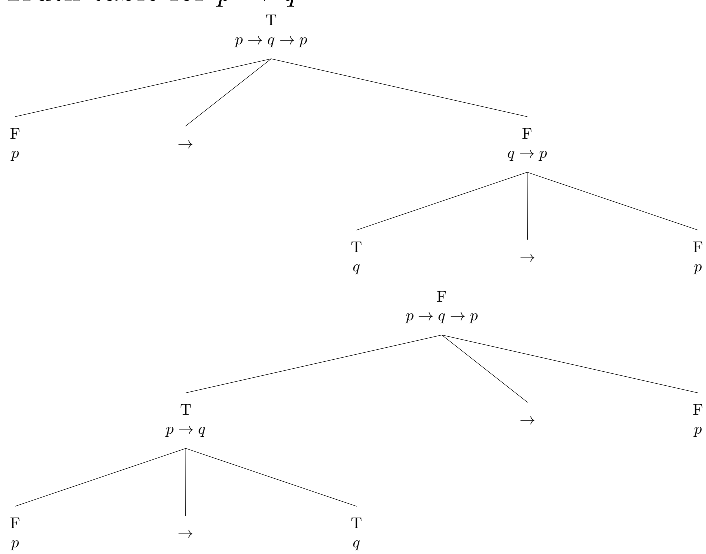


Figure 3: Logic trees for $p \to (q \to p)$ and $(p \to q) \to p$ in w_3

	p	q
w_1	T	T
w_2	T	F
w_3	F	T
w_4	F	F

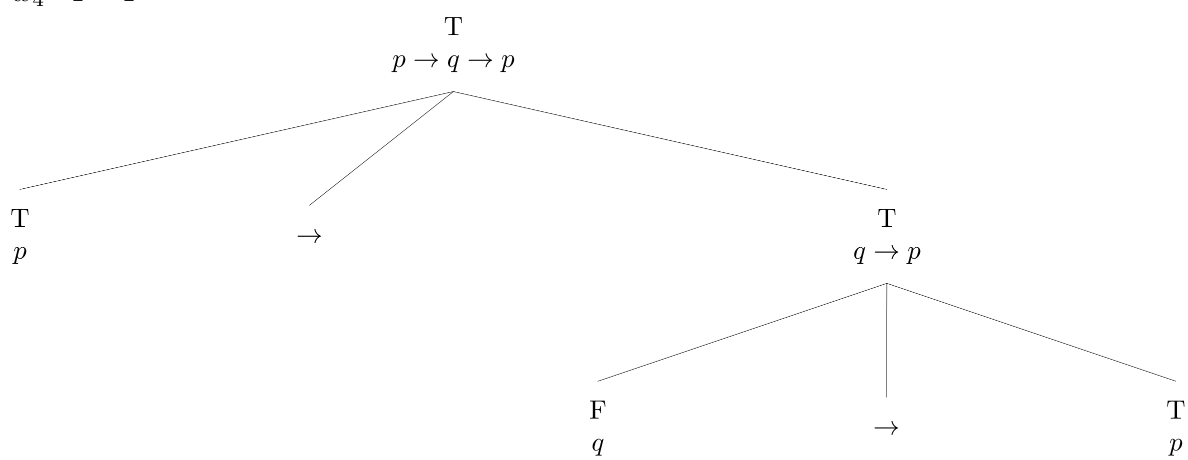


Figure 4: Logic tree for and $p \to (q \to p)$ in w_2

2 Truth tables

Figure 3 shows the logic tree for $(p \to q) \to p$ and $p \to (q \to p)$ in w_3. The truth values of the the two expressions are different in w_3, so we know that these two expressions are not equivalent. There is at least one world in which they differ in truth value. Where you put the parentheses matters.

Figure 4 shows the logic tree for $p \to (q \to p)$ in w_2. This means we know the truth values of $p \to (q \to p)$ when p is F and q is T (w_3) and when p is T and q is F (w_2). So we know two of the four lines for the truth table for $(p \to q) \to p$.

	p	q	$p \to (q \to p)$
(a)	T	T	
(b)	T	F	T
(c)	F	T	T
(d)	F	F	

A truth table computes the truth values for an expression in all possible worlds. But we dont really need to look at all possible worlds to do that. We just need to look at all possible ways of assigning truth values to the sentential letters in the expressions (p and q in this case). Since we have 2 letters, there are 4 different ways of assigning the values T and F and so the truth table has 4 rows. If there were 3 letters, there would 8 rows. In general, if there are n letters, a truth table would have 2^n rows, so they can get quite long.

To finish our truth table, we need to deal with the cases where p and q have the same truth values, that is where p is T and q is T, the row for w_1, and with the case p is F and q is F, the row for w_4.

	p	q
w_1	T	T
w_2	T	F
w_3	F	T
w_4	F	F

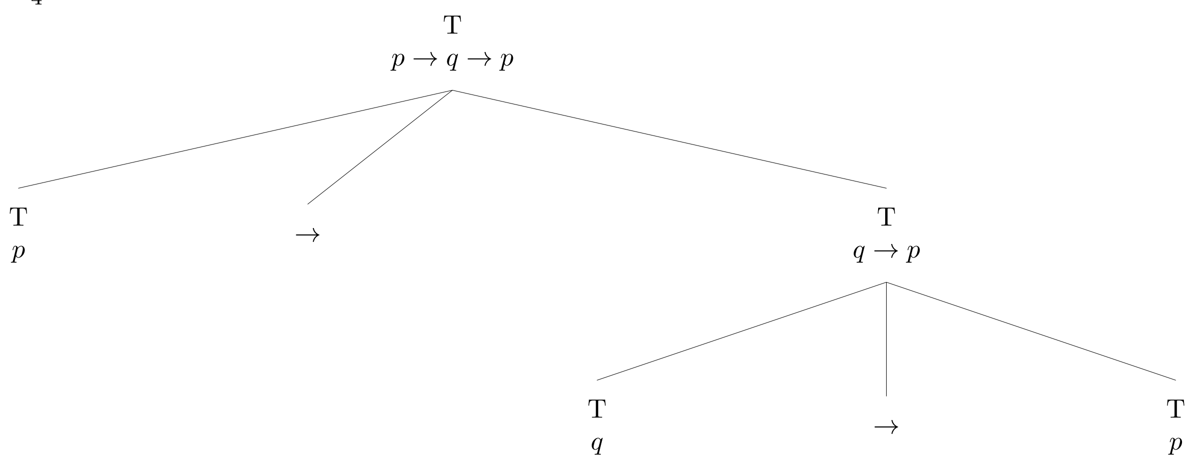


Figure 5: Logic tree for and $p \to (q \to p)$, in w_1 of truth table

	p	q
w_1	T	T
w_2	T	F
w_3	F	T
w_4	F	F

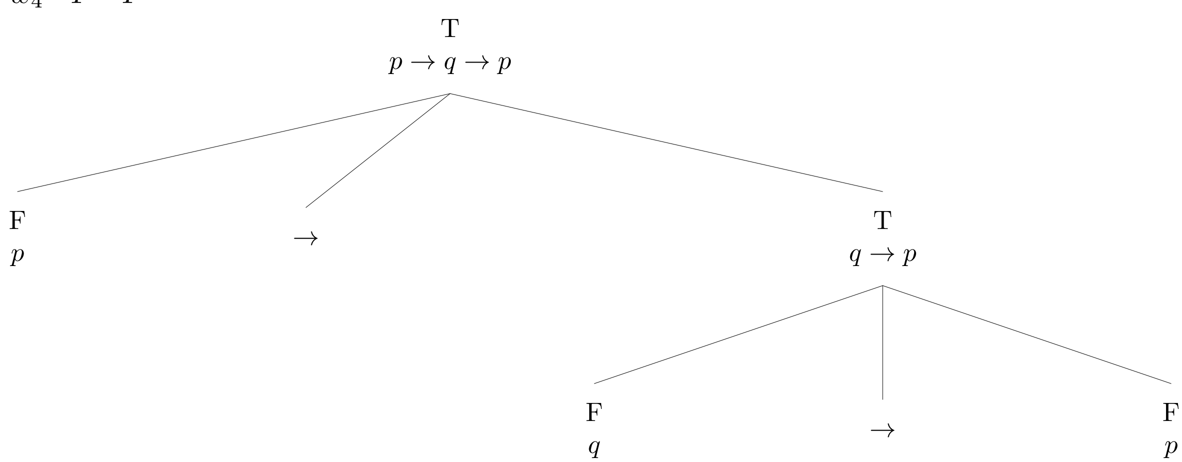


Figure 6: Logic tree for and $p \to (q \to p)$, in w_4 of truth table

Now the completed truth table.

	p	q	$p \to (q \to p)$
(a)	T	T	T
(b)	T	F	T
(c)	F	T	T
(d)	F	F	T

Task: Complete the truth table for $(p \to q) \to p$. Line (c) has already been done for you. (Figure 3)

	p	q	$(p \to q) \to p$
(a)	T	T	
(b)	T	F	
(c)	F	T	F
(d)	F	F	

3 In class exercises

Truth tables: Do complete truth tables for all of the following expressions

In the following exercises, assume the following facts about worlds w_1, w_2, w_3 and w_4.

	p	q	r
w_1	T	T	T
w_2	T	F	F
w_3	F	T	F
w_4	F	F	T

Assume the standard truth tables for & and $\vee$:

p	q	$p \,\&\, q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

1. Draw the logic tree for
$$(p \,\&\, q) \vee r$$
in w_3.

2. Draw the logic tree for
$$(p \,\&\, q) \vee r$$
in w_4.

3. Let:
$$p = \text{John sleeps}$$
$$q = \text{Mary sleeps}$$
Do the logic tree for "John or Mary sleeps" in w_2.

4. Consider "Neither John nor Mary sleeps". Assume
$$p = \text{John sleeps}$$
$$q = \text{Mary sleeps}$$
This is a two-part problem.

   Part A: Using your linguistic intuitions, evaluate the TRUTH of this sentence in worlds w_1 through w_4, and fill in the following table with the results, assuming p is *John sleeps* and q is *Mary sleeps*. I have done the first row for you.

	p	q	Neither p nor q
w_1	T	T	F
w_2	T	F	
w_3	F	T	
w_4	F	F	

   The first row reflects my intuition that "Neither John nor Mary sleeps" is FALSE in a world where "John sleeps" (p) is TRUE and "Mary sleeps" (q) is TRUE.

   Part B: From among the following alternatives, you must choose a logical translation for *Neither John nor Mary sleeps*:

a.	$\sim p \vee q$
b.	$\sim (p \vee q)$
c.	$\sim p \,\&\! \sim q$
d.	$\sim p \;\&\; q$
e.	$p \vee q$
f.	$p \;\&\; q$
g.	$\sim (p \;\&\; n2.disty := 1.2in; q)$

To show that your choice is valid, you must show the entire truth table for your choice.

4 Truth tables and trees: Is "∼" working right?

If p and $\sim p$ are contradictories, they should never **both** be true. In other words:

$$p \;\&\; \sim p$$

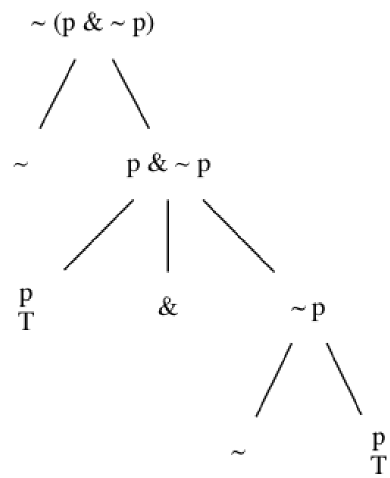
should always be false, whether p is T or F.

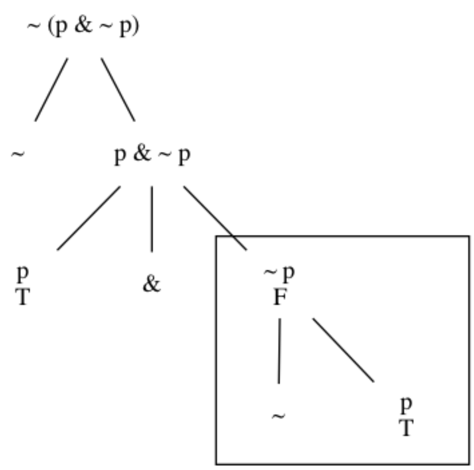
And if **that's** right, then

$$\sim (p \;\&\; \sim p)$$

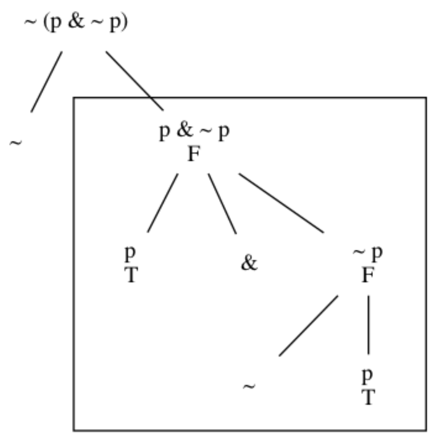
should always be true, whether p is T or F.

Does this turn out to be true?

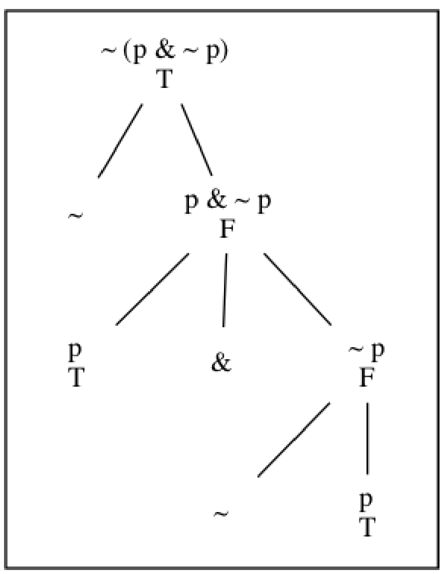




p	$\sim p$
T	**F**
F	T

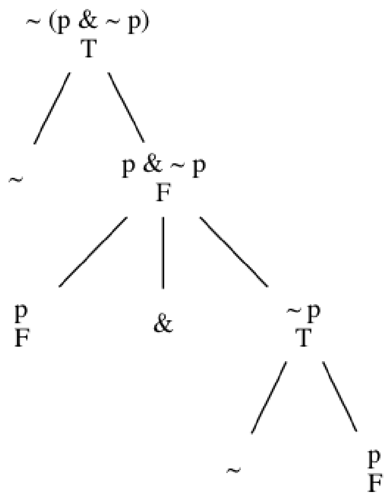


A	B	$A \& B$
T	T	T
T	**F**	**F**
F	T	F
F	F	F



A	$\sim A$
T	F
F	**T**

So far, so good.
Then, following the same method, we try making p false, and we get:



And we're done. No matter what the truth-value of p, $\sim(p\ \&\ \sim p)$ always turns out true. So if our goal is to design a system in which this always turns out true, we've done it.

What do I mean by "a system" here? Well, the system is statement logic. The relevant part of statement logic is the rules for assigning truth values to sentences containing & and $\sim$.

Terminological point: We call a sentence which **always** turns out to be true, no matter what the truth-values of the statements in it, a **valid statement**. Validity is a general concept used in all logical systems. Valid sentences in statement logic are called **tautologies**. We just **proved** $\sim(p\ \&\ \sim p)$ is a tautology.

The following table is called a **truth table**. It summarizes what we did in the two trees above; each row corresponds to one choice of a truth value for p. The last column gives the truth-value for $\sim(p\ \&\ \sim p)$ when p has the given truth value:

p	$\sim p$	$p\ \&\ \sim p$	$\sim(p\ \&\ \sim p)$
T	F	F	T
F	T	F	T

The truth table above tells us $\sim(p\ \&\ \sim p)$ is valid because the last column consists entirely of T's. Now look at the second to last column $p\ \&\ \sim p$. That column is filled entirely with F's. What that tells us is that $p\ \&\ \sim p$ always turns out to be false. Such a statement is called a **contradiction**. So we found our tautology by negating a contradiction. That always works. Find yourself a contradiction and negate it, and you've got a tautology. It works to go the other way too. Negate a tautology and you've got a contradiction. So what is $\sim\sim(p \,\&\, \sim p)$?

We can use a truth table to check the validity of any statement. (call it ϕ and let it consist of atomic statements p, q, ...), The idea is to run through all possible ways of assigning truth values to p, q, ..., and if the last column, which tells us the truth values for ϕ, turns out to consist entirely of trues, then ϕ is a tautology.

Here's another example:

p	q	$p \to q$	$q \to (p \to q)$
T	T	T	?
T	F	F	?
F	T	T	?
F	F	T	?

Try to finish this table without looking at the answer on the next page. Note that the first three columns, filled in, are just the truth table for $\to$ that we used above. To compute the last column, we need to use that truth table, because the main connective of $q \to (p \to q)$ is $\to$.

To fill in the last cell in the first row, look only at truth values in the first row. It tells us that q is true and $p \to q$ is true. So we have:

$$\begin{array}{ccc} q & \to & (p \to q) \\ \mathrm{T} & & \mathrm{T} \end{array}$$

The truth table for $\to$ tells us that when the left hand side and the right hand side are both true, the entire statement is true. So we enter that as the truth value in the last column:

p	q	$p \to q$	$q \to (p \to q)$
T	T	T	T
T	F	F	?
F	T	T	?
F	F	T	?

Let's do the second row. We have.

$$\begin{array}{ccc} q & \rightarrow & (p \rightarrow q) \\ \mathrm{F} & & \mathrm{F} \end{array}$$

The truth table for for $\rightarrow$ tells us that when both the left hand side and the righthand side are false, the entire statement is true (this case is in boldface in the small truth table below). So we enter that in the last column.

p	q	$p \rightarrow q$	$q \rightarrow (p \rightarrow q)$
T	T	T	T
T	**F**	**F**	**T**
F	T	T	?
F	F	T	?

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	**F**	**T**

You should now be able to finish the truth table on your own. The answer is on the next page.

p	q	$p \to q$	$q \to (p \to q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

This too is a tautology. Notice that this time the truth table has four rows because there are four ways of assigning truth values to two distinct statements. How many rows would the truth table for $p \lor q \lor r$ have?

Now try this one all on your own. When you're done, ask your self if the sentence in the last column is a tautology, a contradiction, or neither.

p	$\sim p$	$p \lor \sim p$
T	?	?
F	?	?

Now do this one

p	q	$q \to p$	$(q \to p) \to q$
T	T		
T	F		
F	T		
F	F		

5 Contradictories revisited

Two things we showed in the last section were

1. p and $\sim p$ are contraries. We showed this when we showed that $p \,\&\, \sim p$ is a contradiction. p and $\sim p$ can't both be true.

2. p and $\sim p$ are contradictories. We showed this when we showed that $p \lor \sim p$ is a tautology, That is, either p is true or $\sim p$ is true.

3. The two laws we just proved about the relationship of p and $\sim p$ are important enough to deserve names.

The Law of Contradiction	$\sim (p \mathbin{\&} \sim p)$
The Excluded Middle	$p \vee \sim p$

6 Summary

1. We have truth tables defining a number of connectives: *not* ($\sim$), *and* (&),*or* ($\vee$), and *implies* ($\rightarrow$)
2. Taken collectively these define **statement logic**.
3. A tautology is a statement whose truth table always turns out true.
4. A contradiction is a statement whose truth always turns out false.
5. In statement logic, **by definition**, $\sim p$ is the contradictory of p. We have proved this using truth tables by showing p and $\sim p$ can't both be true (the Law of Contradiction), and either p or $\sim p$ is true (the Excluded Middle.

7 Statement Logic Classification of sentences

7.1 Logical Equivalence

(1) a. $p \rightarrow q$
b. $\sim p \vee q$
c.

	p	q	$\sim p$	$p \rightarrow q$	$\sim p \vee q$
(a)	T	T	F	T	T
(b)	T	F	F	F	F
(c)	F	T	T	T	T
(d)	F	F	T	T	T

d. $(\sim p \vee q) \Longleftrightarrow (p \rightarrow q)$

7.2 Logical truths (Tautologies)

The statement $p \to (q \to p)$, parenthesized THIS way, is always true. Such statements are called **tautologies**:

$p \to q$ rule

	p	q	$p \to q$
(a)	T	T	T
(b)	T	F	F
(c)	F	T	T
(d)	F	F	T

Example using only $p \to q$ rule

	p	q	$q \to p$	$p \to (q \to p)$
(a)	T	T	T	T
(b)	T	F	T	T
(c)	F	T	F	T
(d)	F	F	T	T

There are also English sentences that can't help but be true.

(2) a. It's raining or it's not raining.
b. Every Italian violinist is Italian.
c. Every Italian violinist is a violinist.
d. This yellow pencil is yellow.

Logicians call example (a) a **tautology** because its truth can be shown in a truth table. That's because it has to be true in virtue of the meaning of *and* and *not*, which are connectives in statement logic. The other examples are not like that. The reasons they have to be true is more complicated. Nevertheless they are called **logical truths** because they can proved in a kind of logic called **predicate logic**, which we'll look at later. In the meantime, note that the syntactic form of these sentences has something to do with why they have to be true, but that's not the whole story. Example (2c) above has the same syntactic form as

(3) Every fake gun is a gun.

but (3) is false. Maybe a fake gun is gunlike or gun-ish or similar to a gun, but it's not actually a gun. If it were, it wouldn't be fake.

8 Contingent sentences

Sentences like $(p \to q) \to p$, parenthesized THIS way, are called contingent sentences, They are sometimes true and sometimes false.

	p	q	$p \to q$	$(p \to q) \to p$
(a)	T	T	T	T
(b)	T	F	F	T
(c)	F	T	T	F
(d)	F	F	T	F

Another example:

	p	q	$p \to q$	$p\&(p \to q)$	$\sim q$	$(p\ \&\ (p \to q)) \to \sim q$
(a)	T	T	T	T	F	F
(b)	T	F	F	F	T	T
(c)	F	T	T	F	F	T
(d)	F	F	T	F	T	T

Most English sentences are contingent.

(4) a. Every Italian violinist is temperamental
b. It's raining.
c. This yellow pencil is mine.

8.1 Contradictions

Some sentences have truth tables that always make them false. Such sentences are called **contradictions**, because they can't help but be false:

	p	$\sim p$	$p\& \sim p$
(a)	T	F	F
(b)	F	T	F

Another example:

	p	q	$p \to q$	$\sim (p \to q)$	$q\& \sim (p \to q)$
(a)	T	T	T	F	F
(b)	T	F	F	T	F
(c)	F	T	T	F	F
(d)	F	F	T	F	F

There are also English sentences that can't help but be false. These too are called contradictions.

(5) a. Some Italian violinist is not a violinist.
b. It's raining and it's not raining.
c. This yellow pencil is not yellow.
d. That triangle has four sides.

Notice that some of these contradictions are just the negations of logical truths [(a) and (c)]. Example (b) is the conjunction of contradictories. And example (d) asserts contrary properties of a single entity. So it's easy to generate contradictions of your own, and impress your friends.

8.2 Logical entailment

The statement p & q entails the statement $p \vee q$. If the first is true, then obviously the second has to be true. We can show this by looking at the truth table for both:

	p	q	$p \,\&\, q$	$p \vee q$
(a)	T	T	**T**	T
(b)	T	F	F	T
(c)	F	T	F	T
(d)	F	F	F	F

Checking each row: Whenever $p \,\&\, q$ is true (boldfaced cell in row a), $p \vee q$ is true. We write this as follows:

$$p \,\&\, q \Rightarrow p \vee q$$

Nevertheless, when it comes to English sentences using the word *or*, we need to account for the fact that *or* often communicates exclusivity (either one or the other, but not both). We will call this an implicature. Here are some examples of how the claiom works.

(6) a. Either he rented either a mid-size car or he rented an economy car. [$p \vee q$ translation claims: If in fact he rented both, this is still literally true; but by implicature, he can't have rented both.]

b. Either there's no bathroom in this house or it's on the second floor. [In fact both statement can't be simultaneously true, but that's not due to the meaning of *or*]

c. You can have either the white one or the red one. [intended meaning: but not both, an implicature]

Exploiting the semantics/pragmatics division of labor. boundary: We always translate 'A or B' as $[\![A]\!] \vee [\![B]\!]$ The *exclusive-or* interpretation is an implicature that holds only in certain contexts. In those contexts C:

$$[\![A]\!] \vee [\![B]\!] \Rightarrow_{\mathrm{C}} \sim ([\![A]\!] \,\&\, [\![B]\!])$$

If add the information that A and B aren't both true to the meaning of *or*, you get an exclusive or reading.

This is a **quantity implicature**. Why?

The idea is the following. If we say $p \vee q$, then we choose not to say $p \,\&\, q$, which is more informative. (The fact that $p \,\&\, q$ is more informative than $p \vee q$ is pretty intutitive, another way of aying this is that p & q entails $p \vee q$ or p & $q \Rightarrow p \vee q$). But if both p and q actually **are** true, and we choose to say only $p \vee q$, that would violative the Maxim of Quantity (be as informative as possible). So saying $p \vee q$ implicates that we have some reason for not saying p & q, most likely that p and q are not both true. It is a defeasible implicature. If the context overrules the implicature, then p & q might be possible. For example,

(7) Students must have a pencil or a pen.

In this case context excludes the implicature *students are not allowed to have both a pencil and a pen.* This is natural: a rule that forbids having both would be *more* restrictive. But notice this peculiar reading is exactly what we would get if *or* meant *exclusive or.*

8.3 Summary

We would like our logical translations to capture the following kinds of semantic facts:

1. Explain entailment patterns.

2. Explain contradictions and sentences pairs athat are contradictories or contraries.

3. Explain tautologies.

4. Account for compositionality: How the individual words and the structural meaning contribute to the meaning as a whole.

9 Doing Semantics: Applying Logical Entailment

We now have ways of writing down and investigating some of our key ideas. In this section we demonstrate this by arguing that if q is a contrary of p, then

(8) q entails $\sim p$

This is an important idea. It gives us lots of simple semantic facts. Once we know that *short* and *tall* are contraries, we know that *John is not short* entails *John is not tall*, and similarly for all other contraries.

We start with the fact that q is some contrary of p, using the definition of a contrary of p:

$$\sim (p \,\&\, q) \tag{1}$$

We want to investigate situations in which (1) is true and q is also true. That is we're interested in situations in which

$$q \,\&\!\sim (p \,\&\, q) \tag{2}$$

is true.

What we want to show is:

$$(q \,\&\!\sim (p \,\&\, q)) \Rightarrow\sim p$$

So we investigate the truth tables for $(q \,\&\!\sim (p \,\&\, q))$ and $\sim p$ to see if the second is sentence is true in every row where the first sentence is. How many rows do we need? Well, we only have two statement letters, so 4.

p	q	$p \& q$	$\sim (p \& q)$	$(q \& \sim (p \& q))$	$\sim p$
T	T	?	?	?	?
T	F	?	?	?	?
F	T	?	?	?	?
F	F	?	?	?	?

So we want to show that whenever there is a T in the second to last column there is also a T in the last column.

We'll do the first column together.

p	q	$p \& q$	$\sim (p \& q)$	$(q \& \sim (p \& q))$	$\sim p$
T	T	T	?	?	?
T	F	F	?	?	?
F	T	F	?	?	?
F	F	F	?	?	?

Now finish this.