Truth Sets, Possible Worlds

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Overview

Outline

The world

Facts: The world is all that is the case. (Wittgenstein)

1. From a birdseye view, all issues are settled about the real world. Either the Patriots win the Superbowl or they don't, possibly because it isn't played because of an asteroid hitting the earth. Either Fido is in the living room or he isn't. Either it's raining or it's not. We call whether it's raining or not an **issue**, and issues settled in the world **facts.**
2. The world is the collection of all facts.

Alternative ways of settling issues

Issues **can** be settled in ways other than they are actually settled. The Patriots might not be in the superbowl. That's a possibility. It's not what **actually** happened, but it **could** have happened. Most issues can be settled in more than one way. We call any possible way of settling an issue a **(possible) state of affairs** or just a state of affairs, for short. So here are two states of affairs:

a. win(Patriots, Conference Championship)
b. not win(Patriots, Conference Championship)

Both are states of affairs. (a) happens to be a fact, the one that's true in the real world; but (b) could have been true.

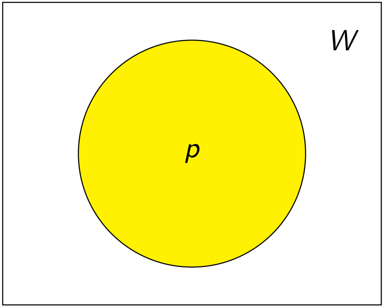
Possible worlds

1. Any collection of states of affairs that settles **all** issues is a **possible world**.
2. The actual world is a possible world.
3. According to the philosopher Leibniz, we live in the **best** of all possible worlds.
4. We will reserve judgment on this (but see *Candide* by Voltaire, for one take on this idea).

Domain of discourse: All possible worlds

It will be useful to use the collection of all possible worlds as a domain of discourse and to pick out **subsets** in which certain states of affairs obtain.

Let p = the set of worlds in which Bruce is a moose

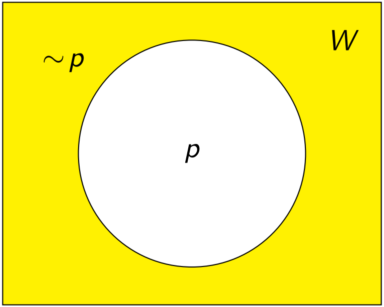


Bruce

No, he isn't

The yellow shaded worlds are those in which Bruce is **not** a moose.

$$\sim p$$



Bruce

A proposition is a truth set

1. We call the claim made by a disambiguated sentence (more on ambiguity elsewhere) the **proposition** it expresses.
2. A proposition is something that can be true or false. Thus, *Bruce* does not express a proposition; *is a moose* does not express a proposition. *Bruce is a moose* expresses a proposition.
3. We call the set of worlds in which some declarative sentence is true its **truth set**. In the previous slides we drew pictures of the truth sets for *Bruce is a moose* and *Bruce is not a moose*.
4. We will identify the truth set of a sentence with the proposition it expresses (though not every one agrees with this idea).
5. Thus the proposition expressed by *Bruce is a moose* is the set of worlds in which Bruce is a moose.

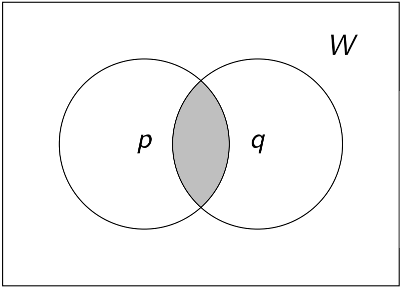
Outline

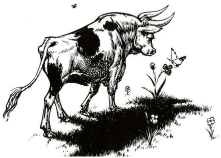
Atomic versus Complex propositions

1. We call a proposition that is not built up out of smaller propositions **atomic**
2. *Bruce is a moose* expresses an atomic proposition. None of the parts of the sentence express propositions.
3. There are ways of combining sentences into more complicated sentences. These express non-atomic or **complex** propositions.
4. But characterizing the exact class of sentences that express atomic propositions is going to be trickier than you might think.

Two propositions

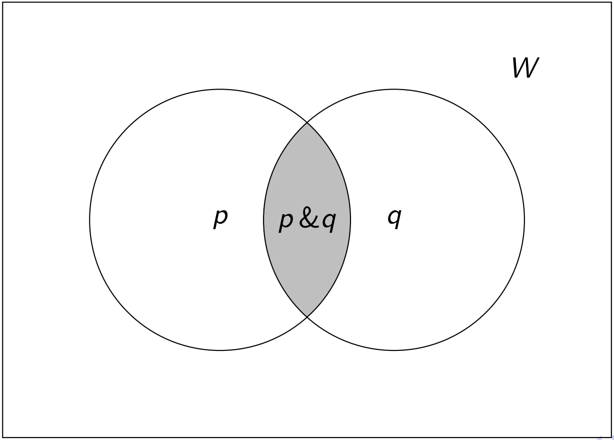
Let p = the set of worlds in which Bruce is a moose
Let q = the set of worlds in which Ferdinand is a bull
What is the shaded area?





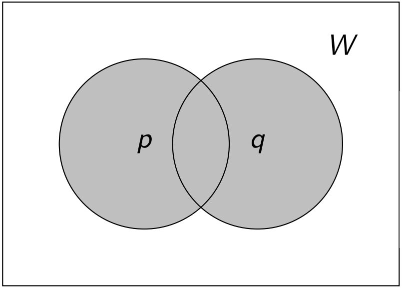
$p \,\&\, q$

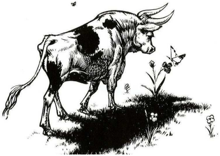
p = worlds in which Bruce is a moose
q = worlds in which Ferdinand is a bull
$p \,\&\, q$ = worlds in which Bruce is a moose and Ferdinand is a bull.



Two propositions again

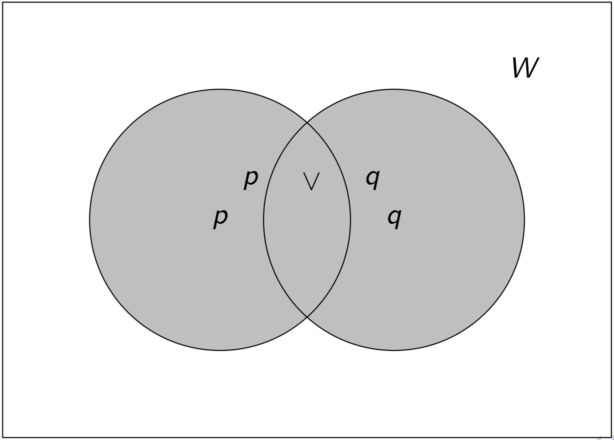
Let p = the set of worlds in which Bruce is a moose
Let q = the set of worlds in which Ferdinand is a bull
What is the shaded area?





$p \vee q$

p = worlds in which Bruce is a moose
q = worlds in which Ferdinand is a bull
$p \vee q$ = worlds in which Bruce is a moose or Ferdinand is a bull.

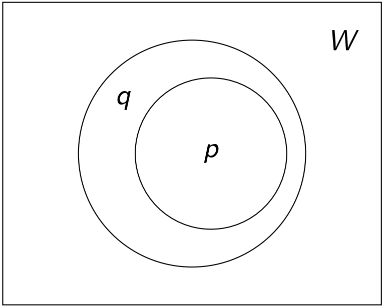


Outline

p entails q

Let p = the set of worlds in which Bruce is a clever moose
Let q = the set of worlds in which Bruce is a moose

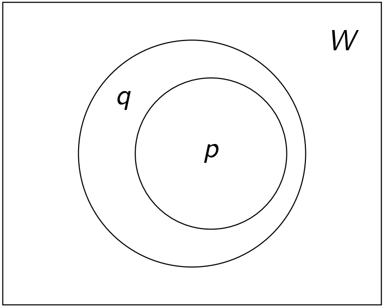
$$p \Rightarrow q$$



Lexical entailment

Let p = the set of worlds in which Bruce is a moose
Let q = the set of worlds in which Bruce is a mammal

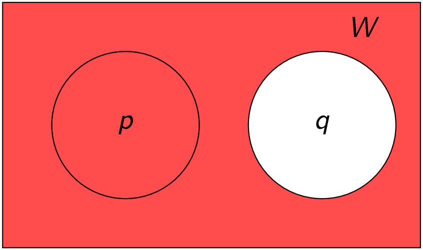
$$p \Rightarrow q$$



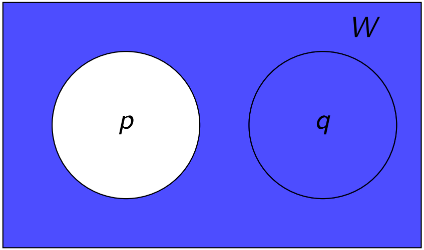
p is a contrary of q

Let p = the set of worlds in which Bruce is a clever moose
Let q = the set of worlds in which Bruce is a dumb moose

$p \Rightarrow \sim q$ (red $= \sim q$)

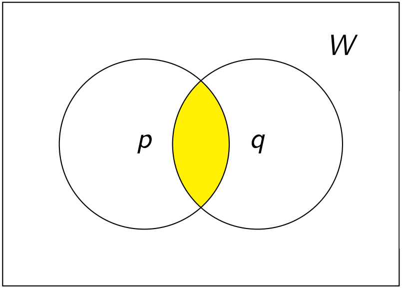


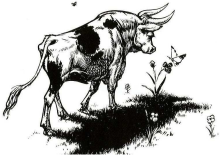
$q \Rightarrow \sim p$ (blue $= \sim p$)



Conjunction entailments

Let p = the set of worlds in which Bruce is a moose
Let q = the set of worlds in which Ferdinand is a bull
$p \;\&\; q \Rightarrow p$ Any world in the shaded area is in the p circle
$p \;\&\; q \Rightarrow q$ Any world in the shaded area is in the q circle

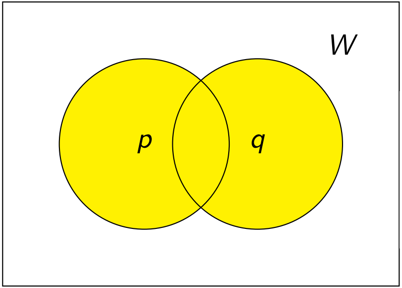


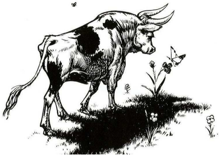


Disjunction entailments?

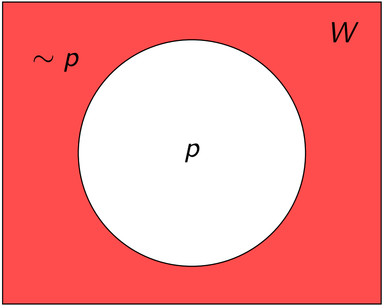
Let p = the set of worlds in which Bruce is a moose
Let q = the set of worlds in which Ferdinand is a bull

$$p \vee q \Rightarrow p\ ? \quad p \Rightarrow p \vee q\ ?$$
$$p \vee q \Rightarrow q\ ? \quad q \Rightarrow p \vee q\ ?$$



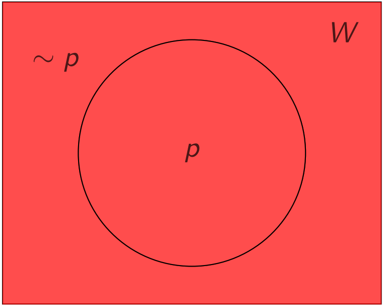


Contradictory



What area would need to be shaded to represent $p \vee \sim p$?

$p \vee \sim p$



$$p \vee \sim p = W$$

That is, the truth set of $p \vee \sim p$ is the set of all possible worlds. In every world, either p is true or $\sim p$ is true. Therefore in every world $p \vee \sim p$ is true.

$\sim p$ is called the **contradictory** of p.

Outline

Truth sets

1. The **truth set** of a sentence is the set of worlds in which the sentence is true. The truth set of *Bruce is a moose* is the set worlds in which Bruce is a moose.
2. We leave open the possibility that two different sentences might have the same truth set. Wouldn't it be great if all such sentences turned out to be synonymous?
3. We use the term truth set and the term **proposition** interchangeably.
4. We will distinguish sentences from their truth sets as follows: *Bruce is a moose* is a sentence. $[\![$*Bruce is a moose*$]\!]$ is the truth set of of *Bruce is a moose*.

Entailment

1. Whenever a sentence p entails a sentence q, we write:

$$p \Rightarrow q$$

2. A sentence p entails another sentence q just in case q has to be true whenever p is true. That is, all possible worlds in which p is true are also worlds in which q is true. That is:

$$[\![\mathrm{p}]\!] \subseteq [\![\mathrm{q}]\!]$$

3. *Bruce is a clever moose* entails *Bruce is a moose.*

$$[\![\text{Bruce is a clever moose}]\!] \subseteq [\![\text{Bruce is a moose}]\!]$$

4. *Bruce is a moose* entails *Bruce is a mammal*

Entailments between complex sentences and simpler sentences

1. *Bruce is a moose and Ferdinand a bull* entails *Bruce is a moose*.
2. *Bruce is a moose and Ferdinand is a bull* also entails *Ferdinand a bull*.
3. *Bruce is a moose* entails *Bruce is a moose or Ferdinand is a bull.*
4. Does *Bruce is a moose and Ferdinand a bull* entail *Bruce is a moose or Ferdinand a bull*? (Think about the Venn diagram). Vice versa?

p = Bruce is a moose.
q = Ferdinand is a bull

$$
\begin{tikzcd}
 & p \,\&\, q \arrow[dl] \arrow[dr] & \\
p \arrow[dr] & & q \arrow[dl] \\
 & p \vee q &
\end{tikzcd}
$$

Contraries and contradictories

1. A sentence p is a **contrary** of another sentence q just in case p and q can't both be true at the same time. That is, all possible worlds in which p is true are also worlds in which q is false. That is:

$$\begin{array}{lll} [\![\mathrm{p}]\!] & \subseteq & [\![\sim q]\!] \\ p & \Rightarrow & \sim q \end{array}$$

2. A sentence p is a **contradictory** of another sentence q just in case p and q can't both be true at the same time, **and** p and q can't both be false at the same time. That is:

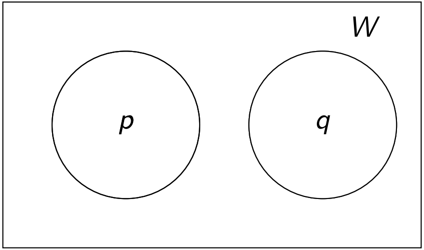
$$\begin{array}{ll} [\![\mathrm{p}]\!] \cap [\![\mathrm{q}]\!] = \emptyset & \text{cant both be true} \\ [\![\mathrm{p}]\!] \cup [\![\mathrm{q}]\!] = W & \text{cant both be false} \end{array}$$

   This definition guarantees that $[\![\mathrm{q}]\!]$ is the **complement** of $[\![\mathrm{p}]\!]$.

3. The contradictory of any sentence p is written $\sim p$.

Examples

1. *Bruce is a dumb moose* and *Bruce is a clever moose* are contraries.
2. *Bruce is a moose* and *Bruce is not a moose* are contradictories.
3. *Bruce is a dumb moose* and *Bruce is a clever moose* are not contradictories. They **can** both be false. Bruce might be the kind of inbetween moose you can't call dumb or smart. This illustrated in our diagram: There are worlds that are neither p-worlds nor q-worlds.



Questions

For each pair of sentences say whether (a) the first entails the second; or (b) is a contrary of the second; or (c) is a contradictory of the second or (d) none of the above.

(1) a. I'm cold.
    b. I'm hot.

(2) a. Hillary Clinton is a former secretary of state.
    b. Hillary Clinton is a secretary of state.

(3) a. Some dogs barked.
    b. Some dogs didn't bark.

(4) a. Every student danced.
    b. Every student didn't dance.

Questions II

(5) a. Every student danced.

b. Not every student danced.

(6) a. Figure A is a triangle.

b. Figure A is a square,

(7) a. Fido didn't bark.

b. Fido barked.

(8) a. Some children went to the park.

b. No children went to the park.

Questions III

(9) a. John sold the book to Mary.
    b. Mary bought the book from John.

(10) a. A San Mateo district attorney brought the case.
    b. The case was brought by a San Mateo district attorney.

(11) a. The case was brought by a San Mateo district attorney.
    b. A San Mateo district attorney brought the case.

Logical Equivalence

Definition

If a sentence p entails a sentence q, and q also entails p, we say p and q are **logically equivalent**, and we write

$$p \Leftrightarrow q$$

For most speakers

(12) a. A San Mateo district attorney brought the case.

b. The case was brought by a San Mateo district attorney.

are logically equivalent.

Question about logical equivalence

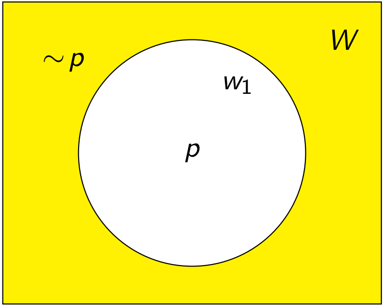
Suppose two sentence p and q are logically equivalent. What can we say about their truth sets?

Outline

Negation truth table

w_1 is a world at which we are evaluating the truth of p and $\sim p$.

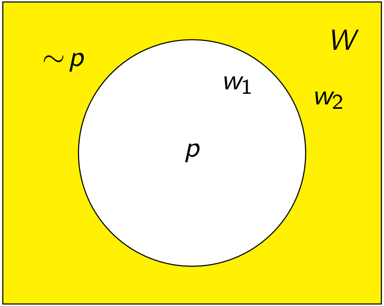
	p	$\sim p$
w_1	T	F



Negation truth table

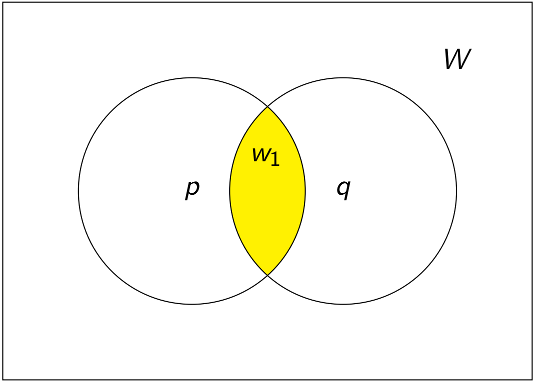
w_2 is another world at which we are evaluating the truth of p and $\sim p$.

	p	$\sim p$
w_1	T	F
w_2	F	T



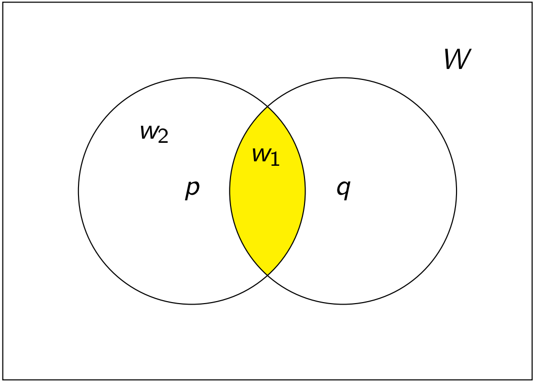
Conjunction, case I

	p	q	$p\ \&\ q$
w_1	T	T	T



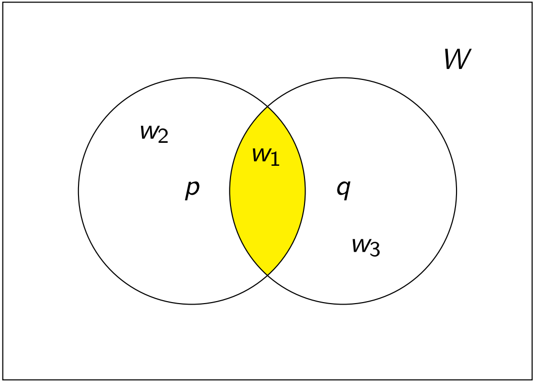
Conjunction, case II

	p	q	p & q
w_1	T	T	T
w_2	T	F	F



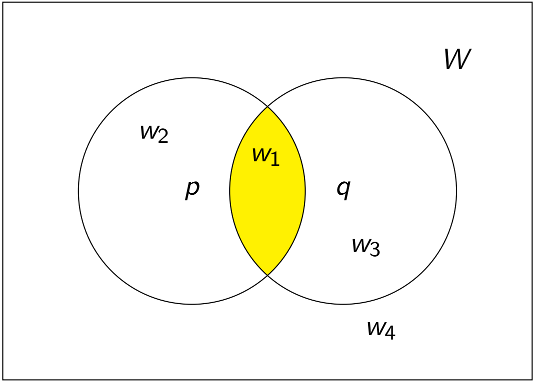
Conjunction, case III

	p	q	$p \,\&\, q$
w_1	T	T	T
w_2	T	F	F
w_3	F	T	F



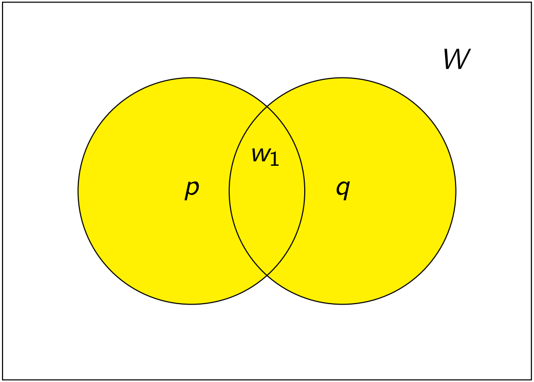
Conjunction, case IV

	p	q	p & q
w_1	T	T	T
w_2	T	F	F
w_3	F	T	F
w_4	F	F	F



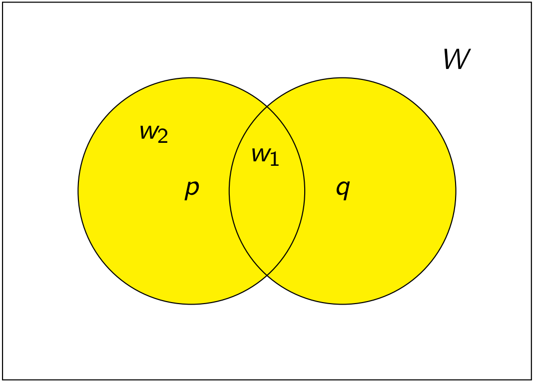
Disjunction, case I

	p	q	$p \vee q$
w_1	T	T	T



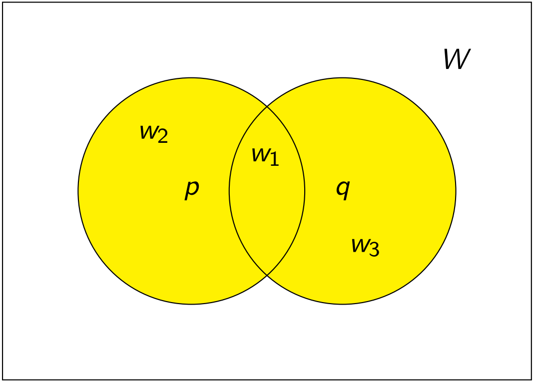
Disjunction, case II

	p	q	$p \vee q$
w_1	T	T	T
w_2	T	F	T



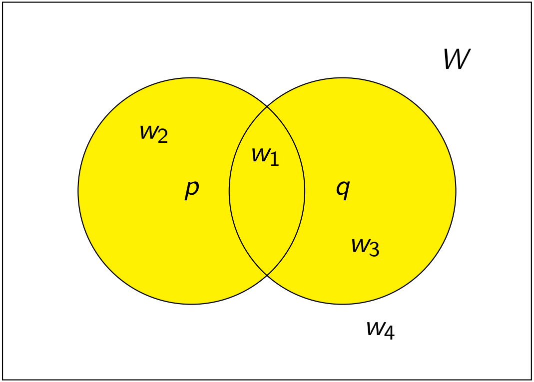
Disjunction, case III

	p	q	$p \vee q$
w_1	T	T	T
w_2	T	F	T
w_3	F	T	T



Disjunction, case IV

	p	q	$p \vee q$
w_1	T	T	T
w_2	T	F	T
w_3	F	T	T
w_4	F	F	F

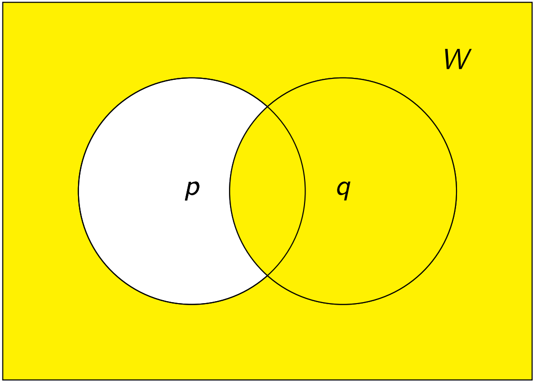


Truth functions: Truth tables

Negation($\sim$)		Conjunction(&)			Disjunction($\vee$)		
p	$\sim p$	p	q	$p \mathbin{\&} q$	p	q	$p \vee q$
T	F	T	T	T	T	T	T
F	T	T	F	F	T	F	T
		F	T	F	F	T	T
		F	F	F	F	F	F

Material implication

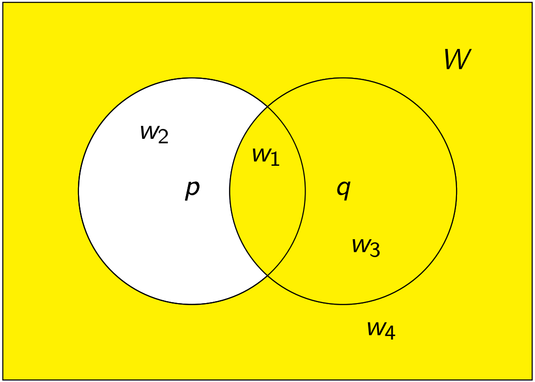
$p \rightarrow q$: p may or may not be true, but if it is, q is also true. (Read this as "p implies q")



Material implication

$p \to q$: p may or may not be true, but if it is, q is also true.

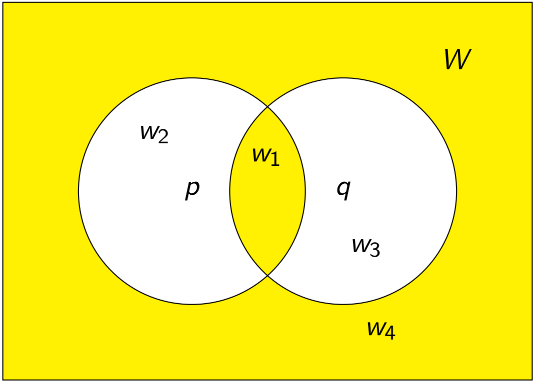
	p	q	$p \to q$
w_1	T	T	T
w_2	T	F	F
w_3	F	T	T
w_4	F	F	T



Material equivalence

$p \equiv q$: p and q are either both true or both false.

	p	q	$p \equiv q$
w_1	T	T	T
w_2	T	F	F
w_3	F	T	F
w_4	F	F	T



Truth functions: Truth tables

Negation($\sim$)		Conjunction(&)			Disjunction($\vee$)			Implication($\rightarrow$)		
p	$\sim p$	p	q	$p \,\&\, q$	p	q	$p \vee q$	p	q	$p \rightarrow q$
T	F	T	T	T	T	T	T	T	T	T
F	T	T	F	F	T	F	T	T	F	F
		F	T	F	F	T	T	F	T	T
		F	F	F	F	F	F	F	F	T

$p \rightarrow q$ vs. $p \Rightarrow q$

1. $p \Rightarrow q$ is a claim about a relationship between the truth sets of the sentences p and q. It says $[\![p]\!]$ is a subset of $[\![q]\!]$. It says: whatever worlds p is true in, q will be true.
2. $p \rightarrow q$ says no such thing. It uses the sentences p and q to make a claim about the facts, not about the sentences. It says it's not the case that p is true and q is false.
3. So "p entails q" ($p \Rightarrow q$) is the sort of thing a linguist might say, talking about a property of two sentences; "p implies q" ($p \rightarrow q$) is said about the world as it is.

Another take

The two diagrams, one for $p \Rightarrow q$, and one for $p \rightarrow q$ both shade the worlds in which $p \rightarrow q$ is true. If $p \Rightarrow q$ is true there **are** are no worlds is which p is true and q is false.

So if $p \Rightarrow q$, then $p \rightarrow q$ is true in all possible worlds.

But they are not the same because $p \rightarrow q$ can be true in in many cases where $p \Rightarrow q$ is false. The second diagram shows that if $[\![p]\!] \not\subseteq [\![q]\!]$, then $p \rightarrow q$ is true only in the shaded worlds.