Truth Sets, Possible Worlds
http://gawron.sdsu.edu/semantics

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Overview

1. Possible Worlds & Propositions
2. Complex propositions
3. Relations between propositions
4. Summarizing proposition relations
5. Truth tables
Outline

1. Possible Worlds & Propositions
2. Complex propositions
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5. Truth tables
Facts: The world is all that is the case. (Wittgenstein)

1. From a birdseye view, all issues are settled about the real world. Either the Patriots win the Superbowl or they don’t, possibly because it isn’t played because of an asteroid hitting the earth. Either Fido is in the living room or he isn’t. Either it’s raining or it’s not. We call whether it’s raining or not an issue, and issues settled in the world facts.

2. The world is the collection of all facts.
Issues **can** be settled in ways other than they are actually settled. The Patriots might not be in the superbowl. That’s a possibility. It’s not what **actually** happened, but it **could** have happened. Most issues can be settled in more than one way. We call any possible way of settling an issue a **(possible) state of affairs** or just a state of affairs, for short. So here are two states of affairs:

a. \( \text{win} (\text{Patriots}, \text{Conference Championship}) \)

b. \( \text{not win} (\text{Patriots}, \text{Conference Championship}) \)

Both are states of affairs. (a) happens to be a fact, the one that’s true in the real world; but (b) could have been true.
Possible worlds

1. Any collection of states of affairs that settles all issues is a possible world.

2. The actual world is a possible world.

3. According to the philosopher Leibniz, we live in the best of all possible worlds.

4. We will reserve judgment on this (but see Candide by Voltaire, for one take on this idea).
Domain of discourse: All possible worlds

It will be useful to use the collection of all possible worlds as a domain of discourse and to pick out **subsets** in which certain states of affairs obtain.

Let $p =$ the set of worlds in which Bruce is a moose
No, he isn’t

The yellow shaded worlds are those in which Bruce is not a moose.

\[ \sim p \]
A proposition is a truth set

1. We call the claim made by a disambiguated sentence (more on ambiguity elsewhere) the **proposition** it expresses.

2. A proposition is something that can be true or false. Thus, *Bruce* does not express a proposition; *is a moose* does not express a proposition. *Bruce is a moose* expresses a proposition.

3. We call the set of worlds in which some declarative sentence is true its **truth set**. In the previous slides we drew pictures of the truth sets for *Bruce is a moose* and *Bruce is not a moose*.

4. We will identify the truth set of a sentence with the proposition it expresses (though not every one agrees with this idea).

5. Thus the proposition expressed by *Bruce is a moose* is the set of worlds in which Bruce is a moose.
Complex propositions

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Atomic versus Complex propositions

1. We call a proposition that is not built up out of smaller propositions atomic.

2. *Bruce is a moose* expresses an atomic proposition. None of the parts of the sentence express propositions.

3. There are ways of combining sentences into more complicated sentences. These express non-atomic or complex propositions.

4. But characterizing the exact class of sentences that express atomic propositions is going to be trickier than you might think.
Two propositions

Let $p$ = the set of worlds in which Bruce is a moose
Let $q$ = the set of worlds in which Ferdinand is a bull
What is the shaded area?
Complex propositions

$p \land q$

$p = \text{worlds in which Bruce is a moose}
q = \text{worlds in which Ferdinand is a bull}

p \land q = \text{worlds in which Bruce is a moose and Ferdinand is a bull.}
Two propositions again

Let $p$ = the set of worlds in which Bruce is a moose
Let $q$ = the set of worlds in which Ferdinand is a bull
What is the shaded area?
$p \lor q$

$p = \text{worlds in which Bruce is a moose}$
$q = \text{worlds in which Ferdinand is a bull}$

$p \lor q = \text{worlds in which Bruce is a moose or Ferdinand is a bull.}$
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1. Possible Worlds & Propositions
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$p$ entails $q$

Let $p =$ the set of worlds in which Bruce is a clever moose
Let $q =$ the set of worlds in which Bruce is a moose

$p \Rightarrow q$

\[
\begin{array}{c}
p \\
q \\
W
\end{array}
\]
Lexical entailment

Let \( p \) = the set of worlds in which Bruce is a moose
Let \( q \) = the set of worlds in which Bruce is a mammal

\[ p \Rightarrow q \]
$p$ is a contrary of $q$

Let $p =$ the set of worlds in which Bruce is a clever moose
Let $q =$ the set of worlds in which Bruce is a dumb moose

$p \Rightarrow \sim q$ (red $= \sim q$)

$q \Rightarrow \sim p$ (blue $= \sim p$)
Conjunction entailments

Let $p = \text{the set of worlds in which Bruce is a moose}$
Let $q = \text{the set of worlds in which Ferdinand is a bull}$

$p \land q \Rightarrow p$ Any world in the shaded area is in the $p$ circle
$p \land q \Rightarrow q$ Any world in the shaded area is in the $q$ circle
Disjunction entailments?

Let \( p = \) the set of worlds in which Bruce is a moose
Let \( q = \) the set of worlds in which Ferdinand is a bull

\[
\begin{align*}
p \lor q & \Rightarrow p \ ? & p & \Rightarrow p \lor q \ ? \\
p \lor q & \Rightarrow q \ ? & q & \Rightarrow p \lor q \ ?
\end{align*}
\]
Contradictory

What area would need to be shaded to represent $p \lor \sim p$?
$p \lor \sim p$}

That is, the truth set of $p \lor \sim p$ is the set of all possible worlds. In every world, either $p$ is true or $\sim p$ is true. Therefore in every world $p \lor \sim p$ is true.

$\sim p$ is called the **contradictory** of $p$. 
Outline

1. Possible Worlds & Propositions
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The truth set of a sentence is the set of worlds in which the sentence is true. The truth set of *Bruce is a moose* is the set worlds in which Bruce is a moose.

We leave open the possibility that two different sentences might have the same truth set. Wouldn’t it be great if all such sentences turned out to be synonymous?

We use the term truth set and the term proposition interchangeably.

We will distinguish sentences from their truth sets as follows: *Bruce is a moose* is a sentence. \([Bruce \textit{ is a moose}]\) is the truth set of of *Bruce is a moose*. 
Entailment

1. Whenever a sentence $p$ entails a sentence $q$, we write:

$$ p \implies q $$

2. A sentence $p$ entails another sentence $q$ just in case $q$ has to be true whenever $p$ is true. That is, all possible worlds in which $p$ is true are also worlds in which $q$ is true. That is:

$$ [p] \subseteq [q] $$

3. *Bruce is a clever moose* entails *Bruce is a moose*.

$$ [\text{Bruce is a clever moose}] \subseteq [\text{Bruce is a moose}] $$

4. *Bruce is a moose* entails *Bruce is a mammal*
Entailments between complex sentences and simpler sentences

1. Bruce is a moose and Ferdinand a bull entails Bruce is a moose.
2. Bruce is a moose and Ferdinand is a bull also entails Ferdinand a bull.
3. Bruce is a moose entails Bruce is a moose or Ferdinand is a bull.
4. Does Bruce is a moose and Ferdinand a bull entail Bruce is a moose or Ferdinand a bull? (Think about the Venn diagram). Vice versa?

\[ p = \text{Bruce is a moose.} \]
\[ q = \text{Ferdinand is a bull} \]

Venn Diagram:

\[ p \quad \lor \quad q \]

\[ p \quad \land \quad q \]
Contraries and contradictories

1. A sentence \( p \) is a **contrary** of another sentence \( q \) just in case \( p \) and \( q \) can’t both be true at the same time. That is, all possible worlds in which \( p \) is true are also worlds in which \( q \) is false. That is:

\[
[f_p] \subseteq [\sim q] \\
p \implies \sim q
\]

2. A sentence \( p \) is a **contradictory** of another sentence \( q \) just in case \( p \) and \( q \) can’t both be true at the same time, **and** \( p \) and \( q \) can’t both be false at the same time. That is:

\[
[f_p] \cap [f_q] = \emptyset \quad \text{cant both be true} \\
[f_p] \cup [f_q] = W \quad \text{cant both be false}
\]

This definition guarantees that \([f_q]\) is the **complement** of \([f_p]\).

3. The contradictory of any sentence \( p \) is written \( \sim p \).
Examples

1. Bruce is a dumb moose and Bruce is a clever moose are contraries.
2. Bruce is a moose and Bruce is not a moose are contradictories.
3. Bruce is a dumb moose and Bruce is a clever moose are not contradictories. They can both be false. Bruce might be the kind of inbetween moose you can’t call dumb or smart. This illustrated in our diagram: There are worlds that are neither p-worlds nor q-worlds.

![Diagram showing p and q circles in a box labeled W]

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Questions

For each pair of sentences say whether (a) the first entails the second; or (b) is a contrary of the second; or (c) is a contradictory of the second or (d) none of the above.

(1) a. I’m cold.
    b. I’m hot.

(2) a. Hillary Clinton is a former secretary of state.
    b. Hillary Clinton is a secretary of state.

(3) a. Some dogs barked.
    b. Some dogs didn’t bark.

(4) a. Every student danced.
    b. Every student didn’t dance.
Questions II

(5) a. Every student danced.
b. Not every student danced.

(6) a. Figure A is a triangle.
b. Figure A is a square,

(7) a. Fido didn’t bark.
b. Fido barked.

(8) a. Some children went to the park.
b. No children went to the park.
Questions III

(9) a. John sold the book to Mary.
   b. Mary bought the book from John.

(10) a. A San Mateo district attorney brought the case.
     b. The case was brought by a San Mateo district attorney.

(11) a. The case was brought by a San Mateo district attorney.
     b. A San Mateo district attorney brought the case.
Logical Equivalence

Definition
If a sentence $p$ entails a sentence $q$, and $q$ also entails $p$, we say $p$ and $q$ are logically equivalent, and we write

$$p \iff q$$

For most speakers

(12) a. A San Mateo district attorney brought the case.
   b. The case was brought by a San Mateo district attorney.

are logically equivalent.
Question about logical equivalence

Suppose two sentences $p$ and $q$ are logically equivalent. What can we say about their truth sets?
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$w_1$ is a world at which we are evaluating the truth of $p$ and $\sim p$.

\[
\begin{array}{c|c}
p & \sim p \\
\hline
w_1 & T & F
\end{array}
\]
Negation truth table

\(w_2\) is another world at which we are evaluating the truth of \(p\) and \(\sim p\).

\[
\begin{array}{c|c|c}
  w & p & \sim p \\
  \hline
  w_1 & T & F \\
  w_2 & F & T \\
\end{array}
\]
Conjunction, case I

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

$W$

$v_1$

$p$

$q$

$W$
### Conjunction, case II

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$p$</th>
<th>$q$</th>
<th>$p &amp; q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

![Venn diagram showing the conjunction of $p$ and $q$.]

The Venn diagram illustrates the truth values for $p$, $q$, and their conjunction $p \& q$. The diagram shows two possible worlds, $w_1$ and $w_2$, with their respective truth values for $p$ and $q$. The conjunction $p \& q$ is true in world $w_1$ and false in world $w_2$. The diagram highlights the overlap and non-overlap regions to represent the truth conditions for the conjunction.
Conjunction, case III

<table>
<thead>
<tr>
<th>( w_1 )</th>
<th>( p )</th>
<th>( q )</th>
<th>( p \land q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
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<tr>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
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<tr>
<td>( F )</td>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
</tbody>
</table>
Conjunction, case IV

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>q</th>
<th>p &amp; q</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$w_2$</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>$w_3$</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>$w_4$</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Disjunction, case I

\[ \begin{array}{ccc}
  w_1 & p & q & p \lor q \\
  T & T & T & T \\
\end{array} \]
### Disjunction, case II

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>$w_2$</th>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
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</table>

####Truth Table

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
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</table>

####Venn Diagram
### Disjunction, case III

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<tbody>
<tr>
<td>( p )</td>
<td>( q )</td>
<td>( p \lor q )</td>
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</tr>
<tr>
<td>( w_1 )</td>
<td>T</td>
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<tr>
<td>( w_2 )</td>
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<td>( w_3 )</td>
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Disjunction, case IV

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<tbody>
<tr>
<td>$p$</td>
<td>$q$</td>
<td>$p \lor q$</td>
<td></td>
</tr>
<tr>
<td>$w_1$</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$w_2$</td>
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<tr>
<td>$w_3$</td>
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<tr>
<td>$w_4$</td>
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</tbody>
</table>
## Truth functions: Truth tables

<table>
<thead>
<tr>
<th>Negation ($\sim$)</th>
<th>Conjunction ($&amp;$)</th>
<th>Disjunction ($\lor$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$\sim p$</td>
<td>$p$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
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</tbody>
</table>
Material implication

$p \rightarrow q$: $p$ may or may not be true, but if it is, $q$ is also true. (Read this as “$p$ implies $q$”)

\[ p \rightarrow q \]

\[ \begin{array}{c|c|c}
  p & q & p \rightarrow q \\
  \hline
  T & T & T \\
  T & F & F \\
  F & T & T \\
  F & F & T \\
\end{array} \]
**Material implication**

$p \rightarrow q$: $p$ may or may not be true, but if it is, $q$ is also true.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
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<tbody>
<tr>
<td>$w_1$</td>
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<tr>
<td>$w_2$</td>
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<tr>
<td>$w_3$</td>
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<td>T</td>
</tr>
<tr>
<td>$w_4$</td>
<td>F</td>
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</tbody>
</table>

[Diagram of possible worlds and implications]

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Gawron: Truth Sets, Possible Worlds

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Material equivalence

\( p \equiv q \): \( p \) and \( q \) are either both true or both false.

\[
\begin{array}{ccc}
| w \ | | p | q | p \equiv q |
|---|---|---|
| w_1 | T | T | T |
| w_2 | T | F | F |
| w_3 | F | T | F |
| w_4 | F | F | T |
\end{array}
\]
### Truth functions: Truth tables

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\sim p$</th>
<th>$p \land q$</th>
<th>$p \lor q$</th>
<th>$p \rightarrow q$</th>
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$p \rightarrow q$ vs. $p \Rightarrow q$

1. $p \Rightarrow q$ is a claim about a relationship between the truth sets of the sentences $p$ and $q$. It says $\llbracket p \rrbracket$ is a subset of $\llbracket q \rrbracket$. It says: whatever worlds $p$ is true in, $q$ will be true.

2. $p \rightarrow q$ says no such thing. It uses the sentences $p$ and $q$ to make a claim about the facts, not about the sentences. It says it’s not the case that $p$ is true and $q$ is false.
Another take

Look at the diagram for when $p \Rightarrow q$ is true. Note that there are no worlds in which $p$ is true and $q$ is false.

So if $p \Rightarrow q$ is true then $p \rightarrow q$ is true in all possible worlds.

But $p \rightarrow q$ can be true in in many case where $p \Rightarrow q$ is false. We’ve already seen that if $[p] \not\subseteq [q]$, as in the second diagram, then $p \rightarrow q$ is true in some worlds and false in others.