Symmetries of the Square

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1 Basic Elements

A square:

```
  D       A
    |
  C       B
```
1.1 Rotations

Rotate it 90° clockwise:

We call this operation R. So:
We call rotating $180^\circ$ clockwise $R'$. So:
We call rotating $270^\circ \ R''$. So:

\[
\begin{array}{cc}
D & A \\
C & B \\
\end{array}
\rightarrow
\begin{array}{cc}
A & B \\
D & C \\
\end{array}
\]

We call rotating $360^\circ \ I$ (for identity). So:

\[
\begin{array}{cc}
D & A \\
C & B \\
\end{array}
\rightarrow
\begin{array}{cc}
D & A \\
C & B \\
\end{array}
\]

That completes the rotations.
1.2 Reflections

Now consider reflections around some axis of the square

There are 4 axes in question. First H and V:

Then D \((x = y)\) and D’ \((x = -y)\):
We illustrate $H$ and $V$:
We illustrate D and D':
1.3 Summarizing Operations

The following square summarizes the rotations:

\[
\begin{array}{ccc}
I & & R'' \\
R & & R' \\
\end{array}
\]

If you start with I on top, then performing an operation puts the side labeled with the operation’s name on top. For example, performing operation R puts R on top.

The following square summarizes the reflections:

\[
\begin{array}{ccc}
V & & D' \\
D & & D' \\
\end{array}
\]

Performing a reflection always flips the square putting back to front. So put this diagram on the back of the square, with V aligned with I. Then if you start with
the I side facing you, with I on top, then performing a particular reflection always ends up with side labeled with that reflection on top.

1.4 Composition

Composing two operations means performing one after another. We write $R \circ H$ for the result of applying $R$ followed by $H$. 
\[ R \circ H = D \]
\[ H \circ R = D' \]
\[ H \circ R \neq R \circ H \]
Identity Element
\[ R \circ R'' = I \]

Inverse Element
What is the inverse of $H$?

$H \circ ? = I$

What is the inverse of $H$?
2 Your own square

Cut out and fold on solid vertical line! After folding D gets superimposed on R”, D’ on R:

Another copy:
Exercises:

1. Compute the following using the square provided.

   \[ R \circ R' \]
   \[ R' \circ R' \]
   \[ H \circ R' \]
   \[ V \circ D \]
   \[ H \circ V \]
3 The Group

The elements of the group are the rotations and reflections. Call this set $G$:

1. $R$: Rotate 90.
2. $R'$: Rotate 180.
3. $R''$: Rotate 270.
4. $I$: Rotate 360.
5. $V$: Reflect around vertical axis.
6. $H$: Reflect around horizontal axis.
7. $D$: Reflect around $x = y$ diagonal.
8. $D'$: Reflect around $x = -y$ diagonal.

The operation is composition ($\circ$).

Theorem 3.1. Grouphood of Symmetries of Square

The set $G$ of rotations and reflections forms a group under the operation of composition.

1. Closure. The composition operation on $G$ is closed. Verify by inspection. Cut out your squares and check. This means filling out the following operation table (also called a Cayley table):

<table>
<thead>
<tr>
<th></th>
<th>$R$</th>
<th>$R'$</th>
<th>$R''$</th>
<th>$I$</th>
<th>$V$</th>
<th>$H$</th>
<th>$D$</th>
<th>$D'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R'$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R''</td>
<td>$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D'$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. **Identity.** \(I\) is the identity element. That it satisfies the identity axiom for groups can be seen by inspecting the following portion of the above table:

<table>
<thead>
<tr>
<th></th>
<th>(R)</th>
<th>(R')</th>
<th>(R'')</th>
<th>(I)</th>
<th>(V)</th>
<th>(H)</th>
<th>(D)</th>
<th>(D')</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(R')</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(R'')</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(I)</td>
<td>(R)</td>
<td>(R')</td>
<td>(R'')</td>
<td>(I)</td>
<td>(V)</td>
<td>(H)</td>
<td>(D)</td>
<td>(D')</td>
</tr>
<tr>
<td>(V)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(H)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(D)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(D')</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. **Inverse.** Each element of \(G\) has an inverse. Verify and show the inverses.

4. **There are three different subgroups having exactly 4 elements.** Find them and draw their group operation tables. Helpful reminder: Remember a subgroup is a group on its own, and must therefore include the identity element and the inverse of every other element.

5. We write \(x^2\) as a shorthand for composing an element with itself, that is \(x \circ x\). For example:

\[
I^2 = I \circ I = I
\]

How many “square roots” does \(I\) have and what are they? Obviously there is at least one, \(I\) itself.

6. **How many subgroups are there having exactly two elements and what are they?** [Hint: having solved the previous problem helps.]