Partially ordered sets and lattices

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Preliminaries
We define an order as a transitive relation $R$ that can be either

1. reflexive and antisymmetric (a weak partial order), written
   $\leq, \subseteq, \sqsubseteq, \ldots$

2. irreflexive and asymmetric (a strong partial order), written
   $\lt, \subset, \subsetneq, \ll, \ldots$
Definition 1. A poset (partially-ordered set) is a set together with a weak order.
Examples of weak orders are $\subseteq$ on sets and $\leq$ on number. The case of $\subseteq$:

<table>
<thead>
<tr>
<th></th>
<th>A $\subseteq$ B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transitive</td>
<td>B $\subseteq$ C</td>
</tr>
<tr>
<td></td>
<td>A $\subseteq$ C</td>
</tr>
<tr>
<td>Reflexive</td>
<td>A $\subseteq$ A</td>
</tr>
<tr>
<td>Antisymmetric</td>
<td>A $\subseteq$ B</td>
</tr>
<tr>
<td></td>
<td>B $\subseteq$ A</td>
</tr>
<tr>
<td></td>
<td>A = B</td>
</tr>
</tbody>
</table>
An English example of a strong order is *ancestor-of*.

<table>
<thead>
<tr>
<th>Property</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transitive</td>
<td>A is the ancestor of B</td>
</tr>
<tr>
<td></td>
<td>B is the ancestor of C</td>
</tr>
<tr>
<td></td>
<td>A is the ancestor of C</td>
</tr>
<tr>
<td>Irreflexive</td>
<td>A is not an ancestor of A</td>
</tr>
<tr>
<td>Asymmetric</td>
<td>A is the ancestor of B</td>
</tr>
<tr>
<td></td>
<td>B is not the ancestor of A</td>
</tr>
<tr>
<td>Type</td>
<td>Example</td>
</tr>
<tr>
<td>-----------</td>
<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Transitive</td>
<td>Lassie is more beautiful than Rin Tin Tin</td>
</tr>
<tr>
<td></td>
<td>Rin Tin Tin is more beautiful than Asta</td>
</tr>
<tr>
<td></td>
<td>Lassie is more beautiful than Asta</td>
</tr>
<tr>
<td>Irreflexive</td>
<td>Lassie is more beautiful than Lassie</td>
</tr>
<tr>
<td>Asymmetric</td>
<td>Lassie is more beautiful than Fido</td>
</tr>
<tr>
<td></td>
<td>Fido is not more beautiful than Lassie</td>
</tr>
</tbody>
</table>
More Examples

1. *is taller than*
2. *is 2 inches taller than?*
3. The dominates relation in trees

```
  M
 /    \
N     O
 |     |
D     H
 |     |
E     I
 |     |
F     J
```
A very simple poset

The diagram of a poset $A = \langle \{\bot, a, b, \top\}, \leq \rangle$
**A poset of sets**

The diagram of the poset $\wp(A)$ for

$$A = \{a, b, c\}$$

A line connecting a lower node to an upper node means the lower node is $\subseteq$ the upper.
Note that not all sets are ordered by $\subseteq$. Thus no line connects $\{a, b\}$ and $\{b, c\}$ because neither is a subset of the other. We say these two sets are incomparable in that ordering.
If an ordering has no incomparable elements, then it is **linear**. For example, the $\leq$ relation on numbers gives a linear ordering.

**Definition 2.** A weak order is linear iff for every pair of elements $a$ and $b$ either $a \leq b$ or $b \leq a$:

$$\forall a, b [a \leq b \lor b \leq a]$$

$\mathbb{N}$ under $<$: A poset with a linear order
$x \leq y$ means $x$ dominates $y$
1. Reflexivity: \( x \leq x \)

2. Transitivity: If \( x \leq y \) and \( y \leq z \) then \( x \leq z \)

3. Antisymmetry: If \( x \leq y \leq x \) then \( x = y \)

4. Single Mother: If \( x \leq z \) and \( y \leq z \) then either \( x \leq y \) or \( y \leq x \) (or both if \( x = y \)).
Reflexivity

M \leq M \quad N \leq N \quad O \leq O
D \leq D \quad E \leq E \quad F \leq F
H \leq H \quad I \leq I \quad J \leq J
Transitivity

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The single mother condition fails!

\[ M_1 \leq O \]
\[ M_2 \leq O \]
\[ \frac{M_1 \leq M_2 \quad \text{No!}}{M_2 \leq M_1 \quad \text{No!}} \]
We say \{a, b\} is a lower bound for \{a, b, c\}.

\{a, b\} \leq \{a, b, c\}

Lower bounds
Lower bounds for sets of things

\( \{a,b,c\} \leq \{a,b,c\} \)

\( \{a,b\} \leq \{a,b\} \)

Note that two sets may share a common lower bound. We generalize the notion lower bound to sets of elements of \( \wp(A) \).
Definition 3. An element of $\mathcal{P}(A)$, $x$, is a lower bound of a subset of $\mathcal{P}(A)$, $S$, if and only if, for every $y \in S$,

$$x \subseteq y$$
Example

\{b\} is a lower bound for \{{a, b}, \{b, c\}\} because

1. \{b\} ⊆ \{a, b\}; and

2. \{b\} ⊆ \{b, c\}.
A set of elements of $\mathcal{P}(A)$ may have multiple lower bounds.

1. The lower bounds of

$$S = \{\{a, b, c\}, \{b, c\}\}$$

are $\{b\}$, $\{c\}$, $\{b, c\}$ and $\emptyset$. There are no others.

2. Of the lower bounds of $S$, $\{b, c\}$ is the greatest lower bound.
Greatest Lower Bound and Least Upper Bound
Definition 4. An element $a$ is the greatest lower bound of a set $S$ (glb of $S$) if and only if:

1. $a$ is a lower bound of $S$
2. For every lower bound $b$ of $S$, $b \leq a$.

In this case we write:

$$a = \text{glb } S$$
Example: greatest lower bounds

1. The lower bounds of

\[ S = \{\{a, b, c\}, \{b, c\}\} \]

are \{b\}, \{b, c\} and \emptyset.

2. Of the lower bounds of \( S \), \{b, c\} is the greatest lower bound.

3. In general, when \( A, B \) are sets,

\[ \text{glb} \{A, B\} = A \cap B \]
Definition 5. An element $a$ is the least upper bound of a set $S$ (*lub of* $S$) if and only if:

1. $a$ is an upper bound of $S$
2. For every upper bound bound $b$ of $S$, $a \leq b$.

*In this case we write:*

$$a = \text{lub } S$$
Example: least upper bounds

1. Within the poset $\varnothing \{a, b, c\}$, the upper bounds of

   $S = \{\{a\}, \{b\}\}$

   are $\{a, b\}$ and $\{a, b, c\}$.

2. Of the upper bounds of $S$, $\{a, b\}$ is the least upper bound.

3. In general, when $A, B$ are sets,

   \[
   \text{lub} = \{A, B\} = A \cup B
   \]
Meet: A greatest lower bound operation

Consider two elements $p$ and $q$ of a poset. We define

\[ p \land q \triangleq \text{glb} \{p, q\} \]

$p \land q$ is read “$p$ meet $q$”. 

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Algebra?

∧ is an operation. Can we use that operation to define an algebra?

1. In an algebra, a two-place operation needs to be defined for every pair of elements in the algebra.
2. In an arbitrary poset the greatest lower bound of a set of elements is not guaranteed to exist.
Join: A least upper bound operation

Consider two elements $p$ and $q$ of a poset. We define

$$p \lor q \triangleq \text{lub} \{p, q\}$$

$p \lor q$ is read “$p$ join $q$”.

In an arbitrary poset the least upper bound of a set is not guaranteed to exist.
LUB and GLB do not always exist

A poset in which least upper and greatest lower bounds do not always exist.
Definition 6. A lattice is a poset $A$ in which, for every $p, q \in A$,

\[
p \land q \in A \\
p \lor q \in A
\]

A lattice poset is an algebra $\langle A, \land, \lor \rangle$ because $\land$ and $\lor$ are operations satisfying the closure and uniqueness requirements (by the above definition).
A very simple lattice poset

The diagram of a simple lattice poset $A = \langle \{\bot, a, b, \top\}, \leq \rangle$
A less simple lattice
1. A **poset** is a set $A$ together with a weak order, written $\leq$. We write this $\langle A, \leq \rangle$, or sometimes just $A$.

2. For any element $x, y \in A$, we say $x$ is a **lower/upper bound** of $y$ iff $x \leq y/y \leq x$.

3. For any subset $S$ of $A$, we say $x$ is a **lower/upper bound of the set** $S$ iff for every $y \in S$, $x \leq y/y \leq x$.

4. For any subset $S$ of $A$, we say $x$ is a **greatest lower/least upper bound of the set** $S$ iff for every lower/upper bound $y$ of $S$, $y \leq x/x \leq y$. We write

$$x = \text{glb } S \quad \text{greatest lower bound}$$

$$x = \text{lub } S \quad \text{least upper bound}$$
The lattice poset of sets

The poset \( \mathcal{P}(A) \) for

\[ A = \{a, b, c\} \]

turns out to be a lattice poset.
Summary: Lattice posets

1. A lattice poset is a poset $A$ in which for any two elements $a$ and $b$,

   $\text{glb} \{a, b\} \in A$
   
   $\text{lub} \{a, b\} \in A$

2. We define two operations $\land$ and $\lor$:

   $a \land b = \text{glb} \{a, b\}$
   
   $a \lor b = \text{lub} \{a, b\}$

3. $\langle A, \land, \lor \rangle$ is an algebra. By definition, $\land$ and $\lor$ satisfy closure and uniqueness.

4. In-class exercise: Prove uniqueness for $\land$ and $\lor$. 