Relations

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### Example 1. English Obstruents

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>Stops</td>
<td>Oral</td>
<td>p/b</td>
<td></td>
<td>t/d</td>
<td></td>
<td></td>
<td></td>
<td>k/g</td>
</tr>
<tr>
<td>Nasal</td>
<td></td>
<td>m</td>
<td></td>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td>η</td>
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<tr>
<td>Fricatives</td>
<td>Oral</td>
<td></td>
<td></td>
<td>f/v</td>
<td>0/ð</td>
<td>s/z</td>
<td></td>
<td>f/ζ</td>
</tr>
</tbody>
</table>
New Sets

Obstruents = \{ p, b, t, d, k, g, m, n, \eta, s, z, \theta, \emptyset, \emptyset, \} | Obstruents| = 16

Stop = \{ p, b, t, d, k, g, m, n, \eta \} | Stops| = 9

Fric = \{ \theta, \emptyset, s, z, \emptyset, \} | Fricatives| = 7

Nas = \{ m, n, \eta \} | Nasals| = 3

Oral = \{ p, b, t, d, s, z, k, g, \theta, \emptyset, \emptyset, \} | Orals| = 13

OralStop = \{ p, b, t, d, k, g \} | OralStops| = 6
Introducing Ordered Pairs

- Ordered Pairs

\[ \langle a, b \rangle \]

- Order matters

\[ \langle a, b \rangle \neq \langle b, a \rangle \]

- Not like sets

\[ \{a, b\} = \{b, a\} \]
Ordered pairs versus sets: More differences

- The same object can occur more than once
  \[ \langle a, a \rangle \neq a \neq \langle a \rangle \neq \{a\} \]

- Unlike sets
  \[ \{a\} = \{a, a\} = \{a, a, a\} \ldots \]
Sets of Ordered pairs [Relations]

- A set of ordered pairs

\[
\text{StopFric} = \{ \langle t, s \rangle, \langle t, z \rangle, \langle d, s \rangle, \langle d, z \rangle, \langle n, s \rangle, \langle n, z \rangle \}\]

- Each *first* member is from Stop, each *second* member is from Fric:

```
    Stop  Fric
     /    /
    /     /
    ⟨t, s⟩ ⟨t, z⟩
```

Relations – p. 6/45
A Relation named \( R \) from \( A \) to \( B \)

\[
R = \{ \langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 3 \rangle, \langle c, 3 \rangle \}
\]
A relation $R'$ from $C$ to $D$ The subsets of $C$ and $D$ actually standing in $R'$ are called the domain and range of $R'$.

$$\text{Dom}(R') = \{a, b, c\}$$

$$\text{Range}(R') = \{1, 2, 3\}$$
Cartesian Products

\[
\text{Stop} \times \text{Fric} = \{ \langle x, y \rangle | x \in \text{Stop} \text{ and } y \in \text{Fric} \}
\]

\[
\{ \langle p, \theta \rangle, \langle p, \bar{\theta} \rangle, \langle p, s \rangle, \langle p, z \rangle, \langle p, \int \rangle, \langle p, 3 \rangle, \langle p, x \rangle, \\
\langle t, \theta \rangle, \langle t, \bar{\theta} \rangle, \langle t, s \rangle, \langle t, z \rangle, \langle t, \int \rangle, \langle t, 3 \rangle, \langle t, x \rangle, \\
\langle k, \theta \rangle, \langle k, \bar{\theta} \rangle, \langle k, s \rangle, \langle k, z \rangle, \langle k, \int \rangle, \langle k, 3 \rangle, \langle k, x \rangle, \\
\langle b, \theta \rangle, \langle b, \bar{\theta} \rangle, \langle b, s \rangle, \langle b, z \rangle, \langle b, \int \rangle, \langle b, 3 \rangle, \langle b, x \rangle, \\
\langle d, \theta \rangle, \langle d, \bar{\theta} \rangle, \langle d, s \rangle, \langle d, z \rangle, \langle d, \int \rangle, \langle d, 3 \rangle, \langle d, x \rangle, \\
\langle g, \theta \rangle, \langle g, \bar{\theta} \rangle, \langle g, s \rangle, \langle g, z \rangle, \langle g, \int \rangle, \langle g, 3 \rangle, \langle g, x \rangle, \\
\langle m, \theta \rangle, \langle m, \bar{\theta} \rangle, \langle m, s \rangle, \langle m, z \rangle, \langle m, \int \rangle, \langle m, 3 \rangle, \langle m, x \rangle, \\
\langle n, \theta \rangle, \langle n, \bar{\theta} \rangle, \langle n, s \rangle, \langle n, z \rangle, \langle n, \int \rangle, \langle n, 3 \rangle, \langle n, x \rangle, \\
\langle \eta, \theta \rangle, \langle \eta, \bar{\theta} \rangle, \langle \eta, s \rangle, \langle \eta, z \rangle, \langle \eta, \int \rangle, \langle \eta, 3 \rangle, \langle \eta, x \rangle \}
\]
Order in Cartesian Products

- Stop × Fric is order dependent, because the ordered-pairs in the set are:

  \[
  \text{Fric} \times \text{Stop} \neq \text{Stop} \times \text{Fric}
  \]

  \[
  \langle s, t \rangle \in \text{Fric} \times \text{Stop}
  \]

  \[
  \langle s, t \rangle \notin \text{Stop} \times \text{Fric}
  \]

  \[
  \langle t, s \rangle \notin \text{Fric} \times \text{Stop}
  \]

  \[
  \langle t, s \rangle \in \text{Stop} \times \text{Fric}
  \]

- Stop × Fric is a special case of a relation from Stop to Fric. For any such relation R,

  \[
  R \subseteq \text{Stop} \times \text{Fric}
  \]
The Cartesian product of $A$ and $B$

$A \times B = \{ \langle a, 1 \rangle, \langle a, 2 \rangle, \langle a, 3 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle b, 3 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle, \langle c, 3 \rangle \}$
Observation: To construct $A \times B$ we need to pair each member of $A$ with all $|B|$ members of $B$. Since we have to do that $|A|$ times

$$|A \times B| = |A| \times |B|$$
A special relation from any set $A$ to itself is the identity relation:

$$I_A = \{ \langle x, x \rangle \mid x \in A \}$$
Relations can be defined by listing OR by predicates, just as sets can be. Reassuring because they are sets.

Example 2. SamePlace

\[ \text{SamePlace} = \{ \langle x, y \rangle \mid \langle x, y \rangle \in \text{Obstruents} \times \text{Obstruents} \text{ and } x \text{ and } y \text{ are articulated in the same place.} \} \]

Example 3. SameVoicing

\[ \text{SameVoicing} = \{ \langle x, y \rangle \mid \langle x, y \rangle \in \text{Obstruents} \times \text{Obstruents} \text{ and } x \text{ and } y \text{ have the same voicing.} \} \]

These are relations from Obstruents to Obstruents.
Example 4. \( \text{StopFric} \)

\[
\text{StopFric} = \{ \langle x, y \rangle \mid \langle x, y \rangle \in \text{Stop} \times \text{Fric} \text{ and } \text{SamePlace}(x, y) \}
\]

Consulting example 1, we find:

\[
\text{StopFric} = \{ \langle t, s \rangle, \langle t, z \rangle, \langle d, s \rangle, \langle d, z \rangle, \langle n, s \rangle, \langle n, z \rangle \}
\]
Definition 1. Inverse of a Relation

We define the inverse $R^{-1}$ or a relation $R$ as follows:

$$R^{-1} = \{ \langle x, y \rangle \mid \langle y, x \rangle \in R \}$$

Note that it is frequently the case that:

$$R^{-1} \cap R = \emptyset$$

When is it not?
A relation and its inverse

Relations – p. 17/45
Example 5. Inverse of StopFric

\[ \text{StopFric}^{-1} = \{ \langle s, t \rangle, \langle z, t \rangle, \langle s, d \rangle, \langle z, d \rangle, \langle s, n \rangle, \langle z, n \rangle \} = \text{FricStop} \]
Definition 2.

\[ R_1 \circ R_2 = \{ \langle x, z \rangle \mid \exists y [\langle x, y \rangle \in R_2 \text{ and } \langle y, z \rangle \in R_1] \} \]

We introduce a new piece of notation here \( \exists \), which means “there exists”.

\[ R_1 \circ R_2 = \{ \langle x, z \rangle \mid \text{There exists a } y \text{ such that } \langle x, y \rangle \in R_2 \text{ and } \langle y, z \rangle \in R_1 \} \]

Note:

If \( R_2 \) is from A to B and \( R_1 \) is from C to D then \( R_1 \circ R_2 \) is from A to D.
Composition of relations $R$ and $S$.

Although $S \circ R$ must be from the same set as $R$ and to the same set as $S$ it does not have to have the same domain as $R$ or the same range as $S$. 
Example 6. Places

Places = \{ bilabial, interdental, alveolar, palatal, velar \}

Example 7. SoundPlace

SoundPlace = \{ \langle x, y \rangle \mid x \in \text{Obstruents} \text{ and } y \in \text{Places} \text{ and } x \text{ is pronounced at place } y \}\}

Example 8. SamePlace defined by composition

SamePlace = \text{SoundPlace}^{-1} \circ \text{SoundPlace}
Example 9. Following the relation links for one pair

\[
\langle t, \text{alveolar} \rangle \in \text{SoundPlace}
\]

Now \(\text{SoundPlace}^{-1} \circ \text{SoundPlace}\) includes all pairs \(\langle t, x \rangle\) such that

\[
\langle \text{alveolar}, x \rangle \in \text{SoundPlace}^{-1}
\]

That is:

\[
\{\langle \text{alveolar}, t \rangle, \langle \text{alveolar}, d \rangle, \langle \text{alveolar}, s \rangle, \langle \text{alveolar}, z \rangle\}
\]

Thus \(\text{SoundPlace}^{-1} \circ \text{SoundPlace}\) includes

\[
\{\langle t, t \rangle, \langle t, d \rangle, \langle t, s \rangle, \langle t, z \rangle\}
\]

Thus for any alveolar sounds \(x\) (including, incidentally, \(t\)):
Functions
Definition 3. Functions

A relation $R$ from $A$ to $B$ is a function from $A$ to $B$ if and only if

1. For each member $x$ of the domain of $R$, there is a unique member of the range standing in relation $R$ to $x$.

2. $\text{Dom}(R) = A$. 

---

A special case of relations
Functions and non functions

F
R
S
Functions and non functions

F is a function from C to D.
R
S
Functions and non functions

F is a function from C to D.
R is not a function from A to B (Clause 2)
S
Functions and non functions

F is a function from C to D.
R is not a function from A to B (Clause 2)
S is not a function (Clause 1)
• **SoundPlace** is a function from obstruents to places because each obstruent has exactly one place it is pronounced at.

• **SoundPlace** is NOT a function because each position may have many sounds pronounced at it. For example

\[
\langle \text{alveolar, } s \rangle \in \text{SoundPlace}^{-1}
\]

\[
\langle \text{alveolar, } t \rangle \in \text{SoundPlace}^{-1}
\]

\[
\langle \text{alveolar, } d \rangle \in \text{SoundPlace}^{-1}
\]

\[
\langle \text{alveolar, } z \rangle \in \text{SoundPlace}^{-1}
\]

\[
\langle \text{alveolar, } n \rangle \in \text{SoundPlace}^{-1}
\]
The successor relation

- The successor relation in arithmetic is a function from integers to integers:

\[
s(0) = 1 \\
s(1) = 2 \\
s(2) = 3 \\
\vdots \\
s(n) = n + 1
\]

- For each integer there is a unique next number. Every integer HAS a successor and every integer has exactly one.
Definition 4. We call a function $F$ from $A$ to $B$ **one-to-one** (or injective) if and only if each member of $F$ is paired with exactly one member of $B$. 
Definition 5. We call a function onto (or surjective) if the range is equal to \( B \). The term into does not mean the opposite of one-to-one. It’s just a synonym for to. A function from \( A \) to \( B \) is also called a function from \( A \) into \( B \), with no implied claim about whether it is onto or not.
One-to-one and onto functions
One-to-one and onto functions

F: 1-1, onto
One-to-one and onto functions

F: 1-1, onto

G: 1-1
One-to-one and onto functions

F: 1-1, onto
G: 1-1
H: neither
One-to-one and onto functions

F: 1-1, onto
G: 1-1
H: neither
I: onto
Some properties of Relations
Reflexive pairs

- We call a pair of the form \( \langle x, x \rangle \) a reflexive pair.
- Note that some relations have reflexive pairs:
  \( \langle t, t \rangle \in \text{same-place} \)
- Some do not:
  \( \langle t, t \rangle \not\in \text{SoundPlace} \)
Definition 6. Reflexivity of a relation
A relation R is reflexive if and only if

\[ \forall x \in \text{Dom}(R) \langle x, x \rangle \in R \]

This introduces the symbol \( \forall \) (upside-down A). \( \forall x \) should be read “for all x”.

for all x such that \( x \in \text{Dom}(R) \langle x, x \rangle \in R \)

\( \forall x \) is always followed by some statement of a condition that all x’s satisfy. Usually we restrict \( \forall x \) to particular set S of x’s: \( \forall x \in S \).
Examples of reflexivity

Example 10. Reflexivity of SamePlace
The SamePlace relation is reflexive. Every sound is pronounced in the same place as itself.

Example 11. NonReflexivity of SoundPlace
The SoundPlace relation is not reflexive. No sound has itself as its place of articulation.
Definition 7. Irreflexivity of a relation
A relation $R$ is irreflexive if and only if

$$\forall x \in \text{Dom}(R) \langle x, x \rangle \not\in R$$
Example 12. *Irreflexivity of SoundPlace*

The SoundPlace relation is irreflexive. No sound has itself as its place of articulation. Observation: Any relation from sets A to set B is trivially irreflexive when A and B are disjoint.
**Neither Reflexive nor Irreflexive**

**Example 13.** *sum-of-factors (perfect numbers)*

Consider the function *sum-factors*.

\[
\text{sum-factors}(x) = y
\]

if and only if the sum of all the prime numbers that evenly divide \( x \), excluding \( x \) equals \( y \).

\[
\begin{align*}
\text{sum-factors}(2) &= 1 \quad 1 \text{ is the only prime factor of 2} \\
\text{sum-factors}(4) &= 3 \quad 1 \text{ and 2 are 4’s prime factors and } 1+2=3
\end{align*}
\]

*Sum-factors is not reflexive. Is it irreflexive? No, because the prime factors of 6 are 1, 2 and 3 and*

\[
\text{sum-factors}(6) = 6 \quad 6 = 1 + 2 + 3
\]
Definition 8. Transitivity of a relation
A relation $R$ is transitive if and only if

$$\forall \langle x, y \rangle \in R \ If \ \exists z \langle y, z \rangle \in R \ then \ \langle x, z \rangle \in R$$
Transitivity: The intuition

Solid links give a partial picture of a relation $R$ from $A$ to $A$. If $R$ is transitive, then when the solid links exist, the dashed links must also exist.
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Examples of transitivity

- The greater-then relation ($<$) on integers is transitive:

  \[ x < y \\
  y < z \\
  \Rightarrow \\
  x < z \]

- The father-of relation is NOT transitive
  Fred is Tom’s father
  Tom is Hank’s father
  \[ \not\Rightarrow \]
  Fred is Hank’s father
  In fact. Fred is Hank’s grandfather
More Transitivity examples

Example 14. Transitivity of brother-of relation. If Fred is Sam’s brother and Alex is Fred’s brother, then Alex is Sam’s brother.

Example 15. Non-Transitivity of brother-in-law relation. If Fred is Sam’s brother-in-law (because Fred is married to Sam’s sister Sue) and Alex is Fred’s brother-in-law (say, because Alex is married to Fred’s sister Frieda), then Alex is not Sam’s brother-in-law.
Non-transitivity of the *brother-in-law* relation. Frieda and Fred are siblings, as are Sue and Sam. Marriage links are labeled $m$; brother-in-law links are labeled $b-i-l$. 
Definition 9. Symmetry of a relation
A relation $R$ is symmetric if and only if

$$\forall \langle x, y \rangle \in R \ [\langle y, x \rangle \in R]$$
Symmetry examples

Example 16. Symmetry of sibling relation
If Fred is Sue’s sibling then Sue is Fred’s sibling.

Example 17. Symmetry of cousin relation
If Fred is Sue’s cousin then Sue is Fred’s cousin.

Example 18. Non-symmetry of brother relation
If Fred is Sue’s brother and Sue is female it is not the case that Sue is Fred’s brother.
Definition 10. Asymmetry of a relation

A relation $R$ is asymmetric if and only if

$$\forall \langle x, y \rangle \in R \ [\langle y, x \rangle \not\in R]$$

Example 19. Asymmetry of father relation.

If Fred is Bob’s father it is definitely not the case that Bob is Fred’s father.

Example 20. Symmetry and the brother relation.

If Fred is someone’s brother they may or may not be his brother, depending on their sex.