Equivalence and Classification

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Equivalence Relations
Definition 1. A relation $R$ in the set $A$ is an equivalence relation in $A$ if and only if

1. $R$ is reflexive on $A$; and
2. $R$ is symmetric; and
3. $R$ is transitive.
Example 1: identity

The identity relation $I_A$ is an equivalence relation on $A$:

- Reflexivity: Everything is equal to itself. For every $x \in A$, $\langle x, x \rangle \in I_A$. 
The identity relation \( I_A \) is an equivalence relation on \( A \):

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- **Symmetry**: Equality is symmetric. If \( x = y \), then \( y = x \).
Example I: identity

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- **Symmetry:** Equality is symmetric. If $x = y$, then $y = x$.

- **Transitivity:** If $x = y$ and $y = z$, then $x = z$. 
Example II: $A \times A$

The Cartesian product of $A$ with itself is an equivalence relation on $A$:

- Reflexivity: For every $x \in A$, $\langle x, x \rangle \in A \times A$. 
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- **Reflexivity:** For every $x \in A$, $\langle x, x \rangle \in A \times A$.
- **Symmetry:** If $\langle x, y \rangle \in A \times A$, this shows $x, y \in A$. So $\langle y, x \rangle \in A \times A$.
- **Transitivity:** If $\langle x, y \rangle \in A \times A$, and $\langle y, z \rangle \in A \times A$, this shows $x, z \in A$. So $\langle x, z \rangle \in A \times A$. 

Example III: Same Voicing, Same Location

• Reflexivity: Every sound has the same voicing as itself.
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Example III: Same Voicing, Same Location

- Reflexivity: Every sound has the same voicing as itself.
- Symmetry: If a sound $x$ has the same voicing as a sound $y$, then $y$ has the same voicing as $x$.
- Transitivity: Suppose $x$ has the same voicing as $y$ and $y$ has the same voicing as $z$, then $x$ has the same voicing as $z$.

Proof: Suppose $x$ is voiced. Then if $x$ stands in the same voicing relation to $y$, then $y$ is voiced. And if $y$ stands in the same voicing relation to $z$, then $z$ is voiced. So $x$ and $z$ have the same voicing. The case of $x$ being voiceless is completely parallel.
Consider this relation \( R \) defined by:

\[
R = \{ (a, b) \mid a - b \text{ is evenly divisible by } 3 \}
\]

A list of numbers \( a \) that stand in the relation \( R \) when \( b = 7 \):

<table>
<thead>
<tr>
<th>( a )</th>
<th>1</th>
<th>4</th>
<th>7</th>
<th>10</th>
<th>13</th>
<th>16</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a - 7 )</td>
<td>-6</td>
<td>-3</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>\ldots</td>
<td></td>
</tr>
</tbody>
</table>

Notice all the numbers in the top row stand in the relation \( R \) to each other, because they are all separated by multiples of 3.

Claim: For any integer \( a \), \( a + m \) stands in the relation \( R \) to \( a \), where \( m \) is a multiple of 3, and all the integers that stand in \( R \) to \( a \) stand in \( R \) to each other.

- Reflexivity: \( x - x = 0 \) and 0 is divisible by 3.
- Symmetry: If \( x - y \) is divisible by 3, then \( y - x \) is divisible by 3.
- Transitivity: Proof left to you.
Non-examples

- Sybling relation
- $\{\langle x, y \rangle \mid \text{height difference of } x \text{ and } y \text{ is less than 5 inches}\}$
- $\{\langle x, y \rangle \mid x \leq y\}$
- $\{\langle x, y \rangle \mid x \text{ and } y \text{ are people who eat at the same restaurant}\}$
If $R$ is an equivalence relation, then when the solid links exist, the links must all exist.

Let $B = \{a, b, c, d, e, f\}$

Then $B \times B \subseteq R$
Equivalence relations: The intuition

If $R$ is an equivalence relation, then when the solid links exist, the dashed links must all exist.

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**Definition 2.** A partitioning of a set $A$ is a set of disjoint sets $p_1, p_2, \ldots p_n$ such that

$$A = p_1 \cup p_2 \cup p_3 \cup \ldots p_n$$

**Definition 3.** An equivalence relation $R$ on $A$ defines a partitioning of $R$. Each of the sets in the partition is called an equivalence class.

We define $\pi_R$, the partition induced by equivalence relation $R$, as follows.

$$\pi_R = \{ S \mid \exists x \in A[S = \{ y \mid \langle x, y \rangle \in R \} ] \}$$
Partitions: The two extreme cases

• $|\pi| = |A|$. The identity relation on a set $A = \{a, b, c\}$ induces the partition:

$$\pi_I = \{ \{a\}, \{b\}, \{c\} \}$$

• $|\pi| = 1$. If $A = \{a, b, c\}$, then $A \times A$ induces the partition:

$$\pi_{A\times A} = \{ \{a, b, c\} \}$$

• Exercise: List the other possible partitions of $A$. Give the relations that induce them.
The difference is divisible by 3 relation partitions the set of integers into three sets we’ll call $S_0$, $S_1$, and $S_2$:

\[
S_0 = \{0, 3, 6, 9, \ldots \} \\
S_1 = \{1, 4, 7, 10, \ldots \} \\
S_2 = \{2, 5, 8, 11, \ldots \}
\]

Note we have 3 disjoint infinite sets which unioned together give us the entire set of integers.
We call the partition induced by the SameVoicing relation **Voice**.
We will call the partition induced by the SamePlace relation **Place**:

\[
\begin{array}{cccc}
p & b & m & f \\
\theta & s & j & k \\
\delta & z & 3 & g \\
\end{array}
\]
Definition 4. A **feature space** is a pair of a set $A$ together with a set $\Pi$ of partitions of $A$. $A$ is called the domain of the feature space. Each member of $\Pi$ is called a **feature**.

Example 1. Let us choose the set of obstruents as our domain, and the partitions induced by the SameVoice relation and the SamePlace relation as our features. Then

$$\text{PhonFeatures} = \langle \text{Obstruents}, \{\text{Place, SameVoice}\} \rangle$$

is a feature space with Obstruents as its domain and

$$\{\text{Place, Voice}\}$$

as its features.
Definition 5. A feature specification $\xi$ chosen from a feature space $F$ is a set $S$ such that each member of $\xi$ is a member of one of the features of $F$.

If

$$\xi = \{ Val1, Val2 \} \quad \text{where} \quad Val1 \in \text{Feat1}, \ Val2 \in \text{Feat2}$$

in feature space

$$F = \langle A, \{ \text{Feat1}, \text{Feat2} \} \rangle,$$

then feature specification $\xi$ can be written

$$\xi = \begin{bmatrix} \text{Feat1} & Val1 \\ \text{Feat2} & Val2 \end{bmatrix}$$

We call $[\xi]$ the denotation of a feature specification, defined as the intersection of its partitions:

$$[\xi] = Val1 \cap Val2$$
Example 2. Let:

\[
\begin{align*}
\text{PhonFeatures} &= \langle \text{Obstruents}, \{\text{Place, Voice}\} \rangle \\
\text{Voice} &= \{ \text{Plus, Minus} \} \\
\text{Place} &= \{ \text{Labial, Labiovelar, Interdental, Alveolar, Alveopalatal, Velar, Glottal} \}
\end{align*}
\]

For example, if:

\[
\begin{align*}
\text{Minus} &= \{ p, f, ð, s, t, k, h \} \\
\text{Alveolar} &= \{ t, d, n, s, z \}
\end{align*}
\]

then

\[
\begin{bmatrix}
\text{Voice} & \text{Minus} \\
\text{Place} & \text{Alveolar}
\end{bmatrix}
\]

is a feature specification denoting \{t, s\}.